Valence independent method for solving string equations

Order N^{-2} equation from $0 = V'(L)_{n,n}$, case $j = 5$:

$$0 = \left(3u^2 + 5z\right) z'' + 4u_2 u^3 + 4uzu'' + 12u^2 z_1 + 6u_1^2 u^2 + 4uu'z' + 2z\left(u'\right)^2 + 12u_1 uz' + 24u_2 uz + 12u_1^2 z + 3\left(z'\right)^2 + 12zz_1$$

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Order N^{-2} equation from $0 = V'(L)_{n,n}$, case $j = 7$:

$$0 = \left(\frac{15u^4}{2} + 75u^2 z + 35z^2\right) z'' + 6u_2 u^5 + 30u^4 z_1 + 15u_1^2 u^4 + 60u_1 u^3 z' + 120u_2 u^3 z + 60uz^2 u'' + 45u^2\left(z'\right)^2 + 180u_1^2 u^2 z + 180u^2 zz_1 + 30z^2\left(u'\right)^2 + 120uzu'z' + 20u^3 zu'' + 20u^3 u'z' + 30u^2 z\left(u'\right)^2 + 180u_2 uz^2 + 90u_1^2 z^2 + 180u_1 uzz' + 60z^2 z_1 + 40z\left(z'\right)^2$$

Order N^{-2} equation from $0 = V'(L)_{n,n}$, variable j:

$$0 = \underbrace{(\sim\sim\sim\sim\sim\sim\sim)z''}_{\text{want formula (general } j\text{)}} + (\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim)$$

Goal: find formula of the following form.

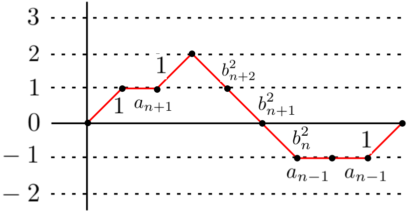
$$(\sim\sim\sim\sim) = P_{\phi,2}^{(a)}(\partial_u, \partial_z) \frac{1}{2\pi i} \oint_0 (h + u + zh^{-1})^{j-1} \frac{dh}{h}$$

Valence independent because

$$\begin{pmatrix} \partial_u^{m+1} \\ \partial_z \partial_u^m \end{pmatrix} \frac{1}{2\pi i} \oint_0 (h + u + zh^{-1})^{j-1} \frac{dh}{h}$$
$$\sim \left[\frac{1}{(z')^2 - z(u')^2} \begin{pmatrix} -zu' & z' \\ zz' & -zu' \end{pmatrix} \partial_x \right]^m \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Order N^{-2} equation of $0 = V'(L)_{n,n}$

$$0 = \left(N^{-2} \text{ term of}\right) \sum_{p \in M_{0,0}^{j-1}} \prod_{i=0}^{j-2} \begin{cases} 1 & \text{if } p(i+1) > p(i) \\ a_{n+p(i)} & \text{if } p(i+1) = p(i) \\ b_{n+p(i)}^2 & \text{if } p(i+1) < p(i) \end{cases}$$



$$
\begin{aligned}
(\sim\sim\sim\sim) &= \sum_{\substack{i_1+j_1+k_1\\ +i_2+j_2+k_2=j-2\\ i_1+i_2-k_1-k_2=1}} u^{j_1+j_2} z^{k_1+k_2} \frac{(i_1-k_1)^2}{2} \binom{i_1+j_1+k_1}{i_1,\ j_1,\ k_1} \binom{i_2+j_2+k_2}{i_2,\ j_2,\ k_2} \\
&= \dots \\
&= \frac{1}{2\pi i} \oint_0 \frac{dh}{h^2} \circ \frac{1}{2\pi i} \oint_0 \frac{d\ell}{\ell^{j-1}} \frac{z^{-1}}{1-\ell(h+u+zh^{-1})} \\
&\qquad \times \frac{(h\partial_h)^2}{2} \frac{1}{1-\ell(h+u+zh^{-1})} \\
&= \dots \\
&= \left(\frac{1}{6}\partial_u^2 + \frac{1}{12}\partial_z\right) \frac{1}{2\pi i} \oint_0 (h+u+zh^{-1})^{j-1} \frac{dh}{h}
\end{aligned}
$$

String polynomials

Theorem. *There exist differential operators* $P^{(a)}_{\lambda,\eta}$, *indexed by integer partitions, so that the string equation* $0 = V'(L)_{n,n}$ *becomes*

$$0 = \sum_{\lambda,\eta} (\partial^{\lambda}_{Nx} a_n)(\partial^{\eta}_{Nx} b_n^2) P^{(a)}_{\lambda,\eta}(\partial_{b_n^2}, \partial_{a_n}) \frac{1}{2\pi i} \oint_0 V'(h + a_n + b_n^2 h^{-1}) \frac{dh}{h}.$$

$$\partial^{\lambda}_{Nx} a_n = \prod_{i=1}^{\mathrm{len}(\lambda)} \frac{\partial_x^{\lambda_i} a_n}{N^{\lambda_i}}$$

A similar formula holds for the string equation $x = V'(L)_{n,n-1}$.

Valence independence

$$\begin{pmatrix} \partial_{a_n}^{m+1} \\ \partial_{b_n^2}\partial_{a_n}^{m} \end{pmatrix} \frac{1}{2\pi i}\oint_0 V'(h + a_n + b_n^2 h^{-1})\frac{dh}{h}$$

$$\sim \left[\frac{1}{(z')^2 - z(u')^2}\begin{pmatrix} -zu' & z' \\ zz' & -zu' \end{pmatrix}\partial_x\right]^m \begin{pmatrix} 0 \\ x \end{pmatrix} + O(N^{-1})$$

String polynomials

Let $l(\lambda)$ and $l(\eta)$ be the numbers of parts of the partitions λ, η. Let I be the left ideal of the ring $\mathbb{C}[B, B^{-1}, \partial_B, \partial_A]$ generated by $\partial_B^2 + B^{-1}\partial_B - 4\partial_A^2$. Then $P^{(a)}_{\lambda,\eta}(\partial_A, \partial_{B^2})$ is the unique element of degree 1 in ∂_{B^2} within the following coset:

$$P^{(a)}_{\lambda,\eta}(\partial_A, \partial_B) \equiv \sum_{\vec{r}} C_{\vec{r}}(\lambda, \eta) W_{\vec{r}}(\partial_A, \partial_B) \partial_A^{l(\lambda)} \partial_{B^2}^{l(\eta)} \qquad \text{mod } I.$$

The coefficients $W_{\vec{r}}$ are defined by

$$\prod_{q=0}^{l(\lambda)+l(\eta)} \left[\ell(h\partial_h)^{r_q}(1 - \ell Bh - \ell A - \ell Bh^{-1})^{-1}\right]$$
$$= W_{\vec{r}}(\partial_A, \partial_B) \begin{cases} \partial_A^{l(\lambda)+l(\eta)}(1 - \ell Bh - \ell A - \ell Bh^{-1})^{-1} & \text{if } \sum r_i \text{ is even} \\ \partial_A^{l(\lambda)+l(\eta)+1} \dfrac{B(h - h^{-1})/l(\eta)}{1 - \ell Bh - \ell A - \ell Bh^{-1}} & \text{if } \sum r_i \text{is odd} \end{cases}$$

The coefficients $C_{\vec{r}}$ are defined by

$$\sum_{m,\sigma} \prod_{q=1}^{l(\lambda)+l(\eta)} \frac{\left(\sum_{r=0}^{q-1} h_r \partial_{h_r} - \sigma_r\right)^{m_q}}{m_q!} = \sum_{\vec{r}} C_{\vec{r}}(\lambda, \eta) \prod_{q=0}^{l(\lambda)+l(\eta)} (h_q \partial_{h_q})^{r_q}.$$

The conditions on the sum over m, σ are that the values of $m : \{1, 2, \ldots, \alpha + \beta\} \to \mathbb{Z}$ are in 1-1 correspondence with the parts of λ and η; and that $\sigma : \{1, 2, \ldots, l(\lambda) + l(\eta)\} \to \{0, 1\}$ is such that if $\sigma(i) = 0$, then $m(i)$ is a part of λ; and if $\sigma(i) = 1$, then $m(i)$ is a part of η.

Table of string polynomials

λ	η	$P^{(a)}_{\lambda,\eta}$	$P^{(b)}_{\lambda,\eta}$
ϕ	ϕ	1	1
1	ϕ	0	$-\frac{1}{2}B^2\partial_{B^2}$
ϕ	1	$\frac{1}{2}\partial_{B^2}$	0
2	ϕ	$\frac{1}{6}B^2\partial_A\partial_{B^2}$	$\frac{1}{6}B^2\partial_A^2 + \frac{1}{12}B^2\partial_{B^2}$
$1+1$	ϕ	$\frac{1}{12}B^2\partial_A^2\partial_{B^2}$	$\frac{1}{12}B^2\partial_A^3 + \frac{1}{12}B^2\partial_A\partial_{B^2}$
1	1	$\frac{1}{6}\partial_A^3$	$\frac{1}{6}B^2\partial_A^2\partial_{B^2}$
ϕ	2	$\frac{1}{6}\partial_A^2 + \frac{1}{12}\partial_{B^2}$	$\frac{1}{6}B^2\partial_A\partial_{B^2}$
ϕ	$1+1$	$\frac{1}{12}B^{-2}\partial_A^2 - \frac{1}{12}B^{-2}\partial_{B^2} + \frac{1}{12}\partial_A^2\partial_{B^2}$	$\frac{1}{12}\partial_A^3 - \frac{1}{12}\partial_A\partial_{B^2}$

What does the valence independent method give us?

z_1 calculation

$$\begin{pmatrix} \phi_m \\ \psi_m \end{pmatrix} = \left[\frac{1}{(z')^2 - z(u')^2} \begin{pmatrix} -zu' & z' \\ zz' & -zu' \end{pmatrix} \partial_x \right]^m \begin{pmatrix} 0 \\ x \end{pmatrix}$$

z_1 calculation

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$$\begin{aligned}
-z_1 =& \psi_3\left(\frac{1}{4}u'\left(z'\right)^2 + \frac{1}{12}z\left(u'\right)^3\right) + \phi_3\left(\frac{1}{4}z\left(u'\right)^2 z' + \frac{\left(z'\right)^3}{12}\right) \\
&+ \psi_2\left(\frac{1}{12}\left(u'\right)^2 z' + \frac{1}{6}zu'u'' + \frac{1}{2}u_1^2 z' - \frac{\left(z'\right)^3}{12z} + \frac{z'z''}{6}\right) \\
&+ \phi_2\left(\frac{1}{6}zu''z' + \frac{1}{2}u_1^2 zu' + \frac{1}{6}zu'z'' + \frac{1}{12}u'\left(z'\right)^2\right) \\
&+ \psi_1\left(\frac{u''z'}{12} + \frac{u'z''}{12} - \frac{u'\left(z'\right)^2}{12z} - \frac{1}{2}u_1'z'\right) \\
=& \dots \\
=& \frac{z}{24}\partial_x^2 \log[(z')^2 - z(u')^2]
\end{aligned}$$

e_1 calculation

$$
\begin{aligned}
e_1 =& \partial_x^{-2} \frac{z_1}{z} - \frac{1}{12} \log \frac{z}{x} \\
=& \frac{1}{24} \log \left[(z')^2 - z(u')^2 \right] - \frac{1}{12} \log \frac{z}{x}
\end{aligned}
$$

Valence independent formulas for e_1 and e_2

$$
\begin{aligned}
e_1 =& -\frac{1}{24}\log z + \frac{1}{24}\log(\alpha'\beta') \\
e_2 =& \frac{1}{23040}\Bigg(-\frac{5\alpha^{(3)}}{\beta'} + \frac{\alpha''\beta''}{(\beta')^2} - \frac{5\beta^{(3)}}{\alpha'} - \frac{9\beta^{(3)}\alpha'}{(\beta')^2} + \frac{15\alpha'\left(\beta''\right)^2}{(\beta')^3} \\
& - \frac{9\alpha^{(3)}\beta'}{(\alpha')^2} - \frac{34\alpha^{(3)}}{\alpha'} + \frac{\alpha''\beta''}{(\alpha')^2} + \frac{15\left(\alpha''\right)^2\beta'}{(\alpha')^3} - \frac{10\alpha''\beta''}{\alpha'\beta'} \\
& + \frac{33\left(\alpha''\right)^2}{(\alpha')^2} - \frac{34\beta^{(3)}}{\beta'} + \frac{33\left(\beta''\right)^2}{(\beta')^2} + \frac{96}{x^2} - \frac{10\alpha''}{\sqrt{z}} - \frac{(\alpha')^3}{4z\beta'} \\
& + \frac{11\alpha'\beta'}{2z} - \frac{(\beta')^3}{4z\alpha'} + \frac{2\alpha'\beta''}{\sqrt{z}\beta'} + \frac{2\beta'\beta''}{\sqrt{z}\alpha'} - \frac{5\left(\alpha'\right)^2}{2z} - \frac{40\alpha^{(4)}\sqrt{z}}{(\alpha')^2} \\
& - \frac{2\alpha'\alpha''}{\sqrt{z}\beta'} - \frac{2\alpha''\beta'}{\sqrt{z}\alpha'} - \frac{128\sqrt{z}\left(\alpha''\right)^3}{(\alpha')^4} + \frac{168\alpha^{(3)}\sqrt{z}\alpha''}{(\alpha')^3} + \frac{10\beta''}{\sqrt{z}} \\
& - \frac{5\left(\beta'\right)^2}{2z} + \frac{40\beta^{(4)}\sqrt{z}}{(\beta')^2} + \frac{128\sqrt{z}\left(\beta''\right)^3}{(\beta')^4} - \frac{168\beta^{(3)}\sqrt{z}\beta''}{(\beta')^3} \Bigg)
\end{aligned}
$$

Theorem. *There exist polynomials P_g such that*

$$z_g = \frac{P_g(u, z, u', z', u'', z'', \ldots)}{\{(z')^2 - z(u')^2\}^{8g-3}}.$$

Summary so far

- We have given an algorithm for computing "valence independent" formulas for e_g.
- These formulas hold for arbitrary polynomial potentials.

Summary so far

- We have given an algorithm for computing “valence independent” formulas for e_g.
- These formulas hold for arbitrary polynomial potentials.
- Can we eliminate the x-derivatives?
- Is there a better way?