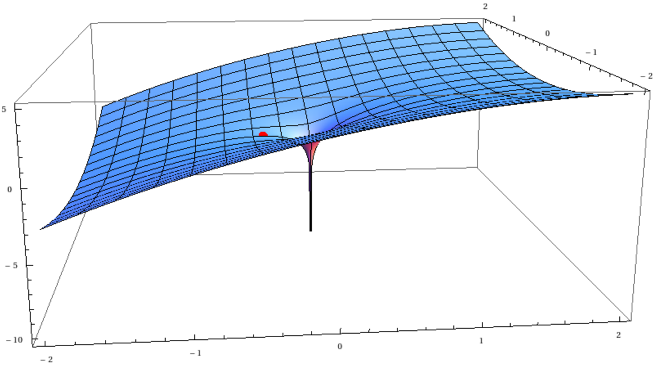
Riemann-Hilbert method for computing e_g

Steepest descent for Hermite polynomials

$$p_n(2\sqrt{n}\lambda) = \frac{n!}{2\pi i} n^{-n/2} \oint_C \exp\left\{-n\left(\log s + 2\lambda s - \frac{1}{2}s^2\right)\right\} \frac{ds}{s}$$



RHP for OP's

$$Y = \begin{pmatrix} p_n & C\{p_n e^{-NV}\} \\ \frac{-2\pi i}{\|p_{n-1}\|^2} p_{n-1} & \frac{-2\pi i}{\|p_{n-1}\|^2} C\{p_{n-1} e^{-NV}\} \end{pmatrix}$$

(C =Cauchy transform)

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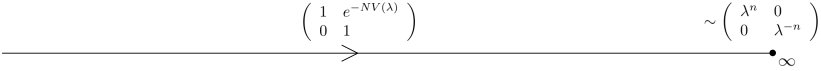
Riemann Hilbert problem (RHP) for Y:



“Steepest descent” for RHP

$$Y = \frac{1}{2}\begin{pmatrix} e^{-Nl/2} & 0 \\ 0 & e^{Nl/2} \end{pmatrix} S$$
$$\times \begin{pmatrix} \gamma + \gamma^{-1} & -i\gamma + i\gamma^{-1} \\ i\gamma - i\gamma^{-1} & \gamma + \gamma^{-1} \end{pmatrix} \begin{pmatrix} e^{Nl/2+ng} & 0 \\ 0 & e^{-Nl/2-ng} \end{pmatrix}$$

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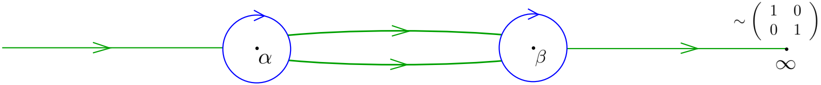
Riemann Hilbert contours for S:



(There is an explicit but complicated formula for the jump matrices along these contours.)

Extracting e_g from the RHP

$$e_1 = C_1 - \frac{1}{6}\log z - \frac{i}{\sqrt{z}}\partial_x^{-1}\left(N^{-1} \text{ term of } \frac{1}{2\pi i}\oint_{\infty} S_{12}(\lambda)\, d\lambda\right)$$

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$$\begin{aligned} e_g = & C_g + \frac{B_{2g}}{(2g)!}\partial_x^{2g-2}\log z \\ & + \sum_{m=0}^{2g-2}\frac{B_m}{m!}\left(N^{m+1-2g} \text{ term of } \log\left\{1 - \frac{z^{-1/2}}{2\pi}\oint_\infty S_{12}(\lambda)\,d\lambda\right\}\right) \end{aligned}$$

e_1 formula from RHP

$$\partial_x e_1 = \frac{2h(\alpha) - 3\sqrt{z}h'(\alpha)}{-48zh(\alpha)^2} + \frac{2h(\beta) + 3\sqrt{z}h'(\beta)}{-48zh(\beta)^2}$$

Where h is the polynomial part of the equilibrium measure:

$$\psi(\lambda) = \frac{1}{2\pi}\sqrt{\lambda - \alpha}\sqrt{\beta - \lambda}h(\lambda)$$

Ok, but can we get an EMcLP-type formula from this?

Valence independent formula for the equilibrium measure

$$h^{(k)}(\beta) = \frac{k!}{x} \sum_{m=1}^{k+1} (2\sqrt{z})^{m-k-1} \left(c_{k,m}^{(\phi)} \phi_m + c_{k,m}^{(\psi)} z^{-1/2} \psi_m \right)$$

$$\frac{(\xi+1)^{k+1}}{(\xi^2-1)^{k+3/2}} = \sum_{m=0}^{k+1} \partial_\xi^m \frac{c_{k,m}^{(\phi)} - \xi c_{k,m}^{(\psi)}}{\sqrt{\xi^2-1}}$$

$$\begin{pmatrix} \phi_m \\ \psi_m \end{pmatrix} = \left[\frac{1}{(z')^2 - z(u')^2} \begin{pmatrix} -zu' & z' \\ zz' & -zu' \end{pmatrix} \partial_x \right]^m \begin{pmatrix} 0 \\ x \end{pmatrix}$$

Results of RH method

- Same valence independent formulas for e_g are obtained
- Hand calculation of e_1 easier by RH method
- Sharpening the qualitative valence indep. formula obtained from the string equations would require understanding some cancellation in valence indep. formula for eq. measure.