First method for computing e_g:

Orthogonal polynomials
and string equations

OP kernel formula

Orthogonal polynomials: if $m \neq n$ then

$$0 = \int_{\mathbb{R}} p_m(\lambda) p_n(\lambda) e^{-NV(\lambda)} \, d\lambda.$$

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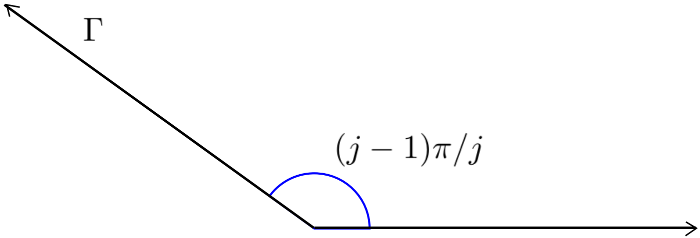
OP kernel formula [1]

$$dP_n(\lambda) = \frac{1}{n!} \det_{i,j=1}^{n} K_{n,N}(\lambda_i, \lambda_j) \, d^n\lambda$$

$$K_{n,N}(\lambda, \eta) = \frac{p_n(\lambda) p_{n-1}(\eta) - p_{n-1}(\lambda) p_n(\eta)}{\|p_{n-1}\|^2 (\lambda - \eta)} \exp\left[-\frac{N}{2} \left(V(\lambda) + V(\eta)\right)\right]$$

OP's for odd-dominant potentials

$$V(\lambda) = \frac{1}{2}\lambda^2 + t\lambda^j, \qquad j \text{ odd, } t > 0.$$



OP's for odd-dominant potentials

Orthogonality

$$0 = \int_{\Gamma} p_m(\lambda) p_n(\lambda) e^{-NV(\lambda)}\, d\lambda$$

Measure defined on

$$\{U\, \mathrm{diag}(\lambda_1, \ldots, \lambda_n) U^{-1} : \ U \text{ Unitary}, \ \lambda_i \in \Gamma\}$$

3-term recurrence

$$\lambda p_n(\lambda) = p_{n+1}(\lambda) + a_n p_n(\lambda) + b_n^2 p_{n-1}(\lambda)$$

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$$\begin{aligned} 0 &= V'(L)_{n,n} \\ x &= V'(L)_{n,n-1} \end{aligned}$$

$$L = \begin{pmatrix} a_0 & 1 & 0 & 0 & \\ b_1^2 & a_1 & 1 & 0 & \\ 0 & b_2^2 & a_2 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Szego relation

$$\frac{Z_{n+1}Z_{n-1}}{Z_n^2} = \frac{n+1}{n} b_n^2$$

String equations, $j = 4$

If $V = \frac{1}{2}\lambda^2 + t_4\lambda^4$ then

$$x = b_n^2 + 4t_4\left(b_{n+1}^2 b_n^2 + b_n^4 + b_n^2 b_{n-1}^2\right)$$

String equations, $j = 3$

If $V = \frac{1}{2}\lambda^2 + t_3\lambda^3$ then

$$
\begin{aligned}
0 =& a_n + 3t_3\left(a_n^2 + b_n^2 + b_{n+1}^2\right) \\
x =& b_n^2 + 3t_3\left(a_{n-1}b_n^2 + a_n b_n^2\right)
\end{aligned}
$$

String equations, $j = 7$

If $V = \frac{1}{2}\lambda^2 + t_7\lambda^7$ then

$$
\begin{aligned}
0 = & a_n + 7t_7\big(a_n^6 + 5b_n^2a_n^4 + 5b_{n+1}^2a_n^4 + 4a_{n-1}b_n^2a_n^3 + 3a_{n+1}^2b_{n+1}^2a_n^2 \\
& + 4a_{n+1}b_{n+1}^2a_n^3 + 6b_n^4a_n^2 + 6b_{n+1}^4a_n^2 + 3a_{n-1}^2b_n^2a_n^2 + 3b_{n-1}^2b_n^2a_n^2 \\
& + 12b_n^2b_{n+1}^2a_n^2 + 3b_{n+1}^2b_{n+2}^2a_n^2 + 6a_{n-1}b_n^4a_n + 6a_{n+1}b_{n+1}^4a_n \\
& + 2a_{n-1}^3b_n^2a_n + 2a_{n-2}b_{n-1}^2b_n^2a_n + 4a_{n-1}b_{n-1}^2b_n^2a_n + 2a_{n+1}^3b_{n+1}^2a_n \\
& + 6a_{n-1}b_n^2b_{n+1}^2a_n + 6a_{n+1}b_n^2b_{n+1}^2a_n + 4a_{n+1}b_{n+1}^2b_{n+2}^2a_n \\
& + b_n^6 + b_{n+1}^6 + 3a_{n-1}^2b_n^4 + 2b_{n-1}^2b_n^4 + 3a_{n+1}^2b_{n+1}^4 + 3b_n^2b_{n+1}^4 \\
& + b_{n+1}^2b_{n+2}^4 + a_{n-1}^4b_n^2 + b_{n-1}^4b_n^2 + a_{n-2}^2b_{n-1}^2b_n^2 + 3a_{n-1}^2b_{n-1}^2b_n^2 \\
& + b_{n-2}^2b_{n-1}^2b_n^2 + 2a_{n-2}a_{n-1}b_{n-1}^2b_n^2 + a_{n+1}^4b_{n+1}^2 + 3b_n^4b_{n+1}^2 \\
& + 2a_{n-1}^2b_n^2b_{n+1}^2 + 2a_{n+1}^2b_n^2b_{n+1}^2 + 2b_{n-1}^2b_n^2b_{n+1}^2 + 2a_{n-1}a_{n+1}b_n^2b_{n+1}^2 \\
& + 2b_{n+1}^4b_{n+2}^2 + 3a_{n+1}^2b_{n+1}^2b_{n+2}^2 + a_{n+2}^2b_{n+1}^2b_{n+2}^2 + 2b_n^2b_{n+1}^2b_{n+2}^2 \\
& + 2a_{n+1}a_{n+2}b_{n+1}^2b_{n+2}^2 + b_{n+1}^2b_{n+2}^2b_{n+3}^2 + 2a_{n+2}b_{n+1}^2b_{n+2}^2a_n\big)
\end{aligned}
$$

Asymptotics of recurrence coefficients

Theorem of Ercolani, McLaughlin and Pierce [2]

$$\log \frac{Z_{n+k,N}(t)}{Z_{n+k,N}(0)} \sim \exp(kN^{-1}\partial_x) \sum_{g\geq 0} e_g N^{2-2g}$$

$$b_{n+k,N}^2 \sim \exp(kN^{-1}\partial_x) \sum_{g\geq 0} z_g N^{-2g}$$

$$a_{n+k,N} \sim \exp(kN^{-1}\partial_x) \sum_{g\geq 0} u_g N^{-g}$$

(as $n, N \to \infty$ with $n/N = x$ fixed)

Theorem of Ercolani, McLaughlin and Pierce [2]

$$
\begin{aligned}
z :=& \frac{1}{16}(\beta - \alpha)^2 \\
=& \lim_{n\to\infty} b_{n,n}^2 \\
=& \partial_{t_1}^2 e_0
\end{aligned}
$$

Asymptotic Szego relation

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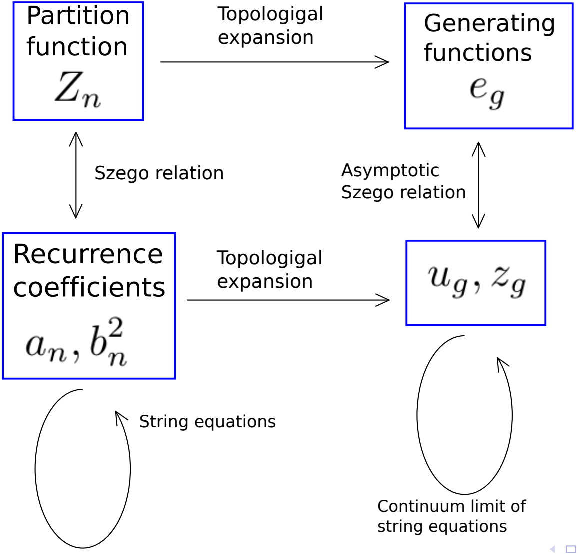
$$e_g = \sum_{m+g'=g} \frac{(1-2m)B_{2m}}{(2m)!} \partial_x^{2m-2} \left(N^{-2g'} \text{ term of } \log b_n^2 \right)$$

$B_m = m^{th}$ Bernoulli number

Calculating e_g by string equations



Odd valences

Why are odd valences harder?

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- If $V(\lambda)$ is even, then $a_n = 0$.

Example: leading order equations

Even valence:

$$x = z + jtz^{j/2}$$

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Odd valence:

$$0 = u + jt \sum_{\mu=0}^{(j-1)/2} \binom{j-1}{2\mu,\ (j-1)2-\mu,\ (j-1)/2-\mu} u^{2\mu} z^{(j-1)/2-\mu}$$

$$1 = z + jt \sum_{\mu=0}^{(j-1)/2} \binom{j-1}{2\mu+1,\ (j-1)2-\mu,\ (j-1)/2-\mu-1} u^{2\mu+1} z^{(j-1)/2-\mu}$$

P. Deift *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, Courant lecture notes in mathematics, 3, NYU, 1999.

N.M. Ercolani, K.D.T.R. McLaughlin, V.U. Pierce, *Random matrices, graphical enumeration and the continuum limit of Toda lattices*. Commun. Math. Phys., 278, 31-81, 2008.