KP equations, Virasoro constraints and the “edge-Toda” equations

KP equation

Kadomtsev-Petviashvili (KP) equation:

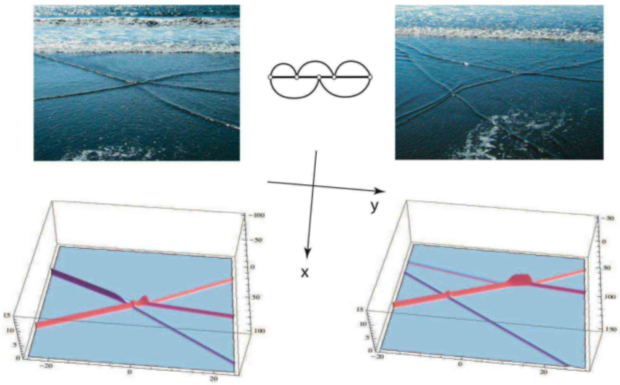
$$0 = 3u_{yy} + \partial_x \left(-4u_t + 12uu_x + N^{-2} u_{xxx} \right)$$

- Models shallow water waves
- Admits *n*-soliton solutions

KP solitons

Image of KP solitons, graphic composed by Yuji Kodama [1]. Photographs by M. Ablowitz.

KP SOLITONS IN SHALLOW WATER



KP and random matrices

Let $t_1 = x, t_2 = y, t_3 = t$ and introduce variables $t_4, t_5, \ldots$

$$
\begin{aligned}
u =& N^{-2}\partial_x^2 \log \tau \\
0 =& \oint_0 e^{N\sum_{i=1}^{\infty}(t_i - s_i)\lambda^i} \tau\left(t_1 - N^{-1}\lambda^{-1}, t_2 - \frac{1}{2}N^{-1}\lambda^{-2}, \ldots\right) \\
& \times \tau\left(s_1 + N^{-1}\lambda^{-1}, s_2 + \frac{1}{2}N^{-1}\lambda^{-2}, \ldots\right) d\lambda
\end{aligned}
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\end{aligned}
$$

Random matrix partition functions are KP τ-functions:

$$
Z_{n,N}(t) = \int_{H(n\times n)} \exp\left[-N\,\mathrm{Tr}\left(\frac{1}{2}M^2 + \sum_{j=1}^{2\nu} t_j M^j\right)\right] dM
$$

RM partition function is a KP tau-function

$$\tau_n(t) = \frac{Z_{n,N}(t)}{Z_{n,N}(0)}$$

Wave function and dual wave function:

$$\Psi_n(\lambda) = \lambda^{-n} p_n(\lambda) e^{N\xi(t,\lambda)}$$

$$\Psi_n^*(\lambda) = \frac{e^{-N\xi(t,\lambda)}}{\|p_{n-1}\|^2} \int \frac{\eta^{n-1} p_{n-1}(\eta)}{1-\eta/\lambda} e^{-NV(\eta)}\, d\eta$$

Virasoro constraints

$$\epsilon \int \left[\mathrm{Tr} \, (I - \epsilon M)^{-1} \right]^2 \, dM = N \int \mathrm{Tr} \, \left[V'(M)(I - \epsilon M)^{-1} \right] \, dM$$

$$0 = N \int \mathrm{Tr} \, V'(M) \, dP_n(M)$$

$$n = N \int \mathrm{Tr} \, V'(M) \mathrm{Tr} \, M \, dP_n(M)$$

Virasoro constraints

Generating function for Virasoro constraints

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Virasoro constraint at order ϵ^0:

$$0 = n^2 + \partial_{t_2} \log Z_{n,N} + \sum_{j\geq 1} j t_j \partial_{t_j} \log Z_{n,N}$$

Edge-Toda equation

$$N^{-1}a_n + N^{-1}\sum_{j\geq 1} jt_j\partial_j a_n = a_{n+1}b_{n+1}^2 - a_n b_n^2 + a_n b_{n+1}^2 - a_{n-1}b_n^2$$

$$N^{-1}2b_n^2 + N^{-1}\sum_{j\geq 1} jt_j\partial_j b_n^2 = b_n^2\left(a_n^2 - a_{n-1}^2 + b_{n+1}^2 - b_{n-1}^2\right).$$

Edge variable

$$dP_n(M) = \frac{1}{Z_n} \exp\left\{ -N \mathrm{Tr} \left(\frac{1}{2q} M^2 + \sum t_j M^j \right) \right\} dM$$

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$$e_g(x, q, t) = \sum_{\Gamma \in \text{ genus } g \text{ maps}} q^{E(\Gamma)} x^{F(\Gamma)} \prod_j \frac{(-t_j)^{V_j(\Gamma)}}{V_j(\Gamma)!}$$

Applications of the edge-Toda equation

Leading order of edge-Toda equation

j-regular case:

$$\begin{aligned} 0 =& \alpha + jt\alpha_t + \frac{1}{8}(3\alpha^2 - 2\alpha\beta - \beta^2)\alpha_x \\ 0 =& \beta + jt\beta_t + \frac{1}{8}(3\beta^2 - 2\alpha\beta - \alpha^2)\beta_x \end{aligned}$$

Leading order of edge-Toda equation

j-regular case:

$$0 = \alpha + jt\alpha_t + \frac{1}{8}(3\alpha^2 - 2\alpha\beta - \beta^2)\alpha_x$$
$$0 = \beta + jt\beta_t + \frac{1}{8}(3\beta^2 - 2\alpha\beta - \alpha^2)\beta_x$$

Similarity reduction

$$\alpha' = \frac{\alpha}{jx - (j-2)(3\alpha^2 - 2\alpha\beta - \beta^2)/16}$$
$$\beta' = \frac{\beta}{jx - (j-2)(3\beta^2 - 2\alpha\beta - \alpha^2)/16}.$$

e_0 formula

$$u = \frac{1}{2}(\alpha + \beta) \qquad z = \frac{1}{16}(\beta - \alpha)^2$$

$$\partial_x^2 e_0 = \log \frac{z}{x}$$

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$$e_0 = \frac{1}{2}\log z + \frac{(j-2)^2}{4j(j+2)}\left\{(z-1)\left(z - \frac{3j+6}{j-2}\right) + 2u^2\left(z - \frac{j+1}{j-2}\right)\right\}.$$

e_1 formula

$$\begin{aligned} e_1 &= \frac{1}{24} \log \frac{z(u')^2 - (z')^2}{z^2} \\ &= \frac{1}{24} \log \frac{4 - u^2/z}{(jx - (j-2)z)^2 - (j-2)^2 u^2 z} \end{aligned}$$

Integration formula

$$(2g-2)j\partial_x^{-2}N^{-2g} \text{ term of } \log b_n^2 = ?$$

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$$\begin{aligned}
=&\frac{-jx^2}{2g-1}\widetilde{z}_g + \frac{(j-2)x}{4g-2}[N^{-2g}]\sum_{m\geq 0}\frac{(N^{-1}\partial_x)^m}{(m+1)!}(a_{n-1}^2+b_{n-1}^2+b_n^2)\\
&+\frac{(j-2)jx}{2j(2g-1)-4}[N^{-2g}](a_{n-1}^2+b_{n-1}^2+b_n^2)\\
&+\frac{(j-2)^2}{4(2g-1)-8}[N^{-2g}]\sum_{m\geq 0}\frac{(N^{-1}\partial_x)^m}{(m+1)!}\left(b_{n-1}^2(a_{n-2}+a_{n-1})^2+b_{n-2}^2b_{n-1}^2\right)\\
&-\frac{j-2}{2}[N^{1-2g}]\sum_{m\geq 0}\frac{(N^{-1}\partial_x)^m}{(m+2)!}(a_{n-1}^2+b_{n-1}^2+b_n^2).
\end{aligned}$$

Y. Kodama *KP solitons in shallow water*, Journal of Physics A: Mathematical and Theoretical, 43, 434004, 2010.