Perturbed GUE

M is Hermitian
dM is Lebesgue measure

$$dP_n(M) = \frac{1}{Z_n(t)} \exp\left[-n\,\mathrm{tr}\left(\frac{1}{2}M^2 + \sum_{j=1}^{2\nu} t_j M^j\right)\right] dM$$

Partition function

$$Z_n(t) = \int_{H(n\times n)} \exp\left[-n\,\mathrm{tr}\left(\frac{1}{2}M^2 + \sum_{j=1}^{2\nu} t_j M^j\right)\right] dM$$

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$$-\frac{1}{n}\partial_{t_k} \log Z_n(t) = \mathbb{E}\left[\mathrm{tr}\left(M^k\right)\right]$$

Maps and matrix integrals

Partition function for quartic potential:

$$Z_n(t) = \int_{H(n\times n)} \exp\left[-n\,\mathrm{tr}\left(\frac{1}{2}M^2 + tM^4\right)\right] dM$$

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Asymptotic expansion as $n \to \infty$:

$$\log \frac{Z_n(t)}{Z_n(0)} \sim e_0(t)n^2 + e_1(t)n^0 + e_2(t)n^{-2} + \dots$$

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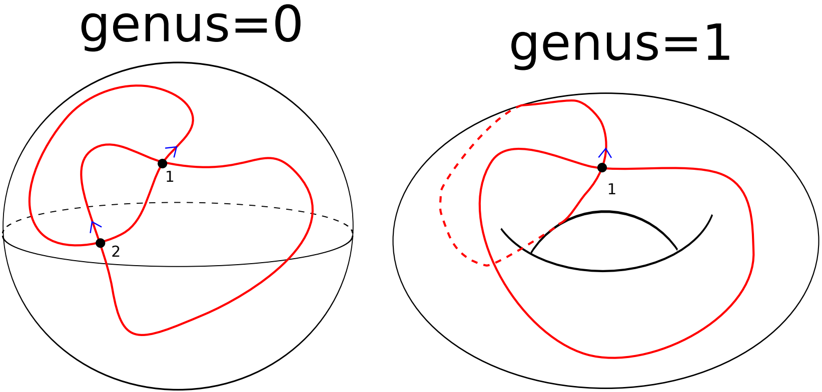
The functions e_g are exponential generating functions for genus g maps with 4 edges meeting at each vertex.

$$e_g(t) = \sum_{k\geq 0} \left(\#\left\{\begin{matrix}k\text{-vertex}\\ 4\text{-valent}\\ g\text{-maps}\end{matrix}\right\}\right) \frac{(-t)^k}{k!}$$

Maps



Faces of maps

Partition function for quartic potential:

$$Z_{n,N}(t) = \int_{H(n\times n)} \exp\left[-N\,\mathrm{tr}\left(\frac{1}{2}M^2 + tM^4\right)\right] dM$$

Faces of maps

Partition function for quartic potential:

$$Z_{n,N}(t) = \int_{H(n\times n)} \exp\left[-N \operatorname{tr}\left(\frac{1}{2}M^2 + tM^4\right)\right] dM$$

Asymptotic expansion as $n, N \to \infty$ with $n/N = x$ fixed:

$$\log \frac{Z_{n,N}(t)}{Z_{n,N}(0)} \sim e_0(x,t)N^2 + e_1(x,t)N^0 + e_2(x,t)N^{-2} + \dots$$

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Generating functions 2

$$Z_{n,N}(t) = \int_{H(n\times n)} \exp\left[-N\mathrm{tr}\left(\frac{1}{2}M^2 + tM^j\right)\right] dM$$

$$e_g(x,t) = \sum_{k\geq 0} x^{\#\text{ faces}} \left(\# \left\{\begin{matrix} k\text{-vertex} \\ j\text{-valent} \\ g\text{-maps} \end{matrix}\right\}\right) \frac{(-t)^k}{k!}$$

$$U(T,\gamma,\nu) = \left\{ (x,t): \ |t_i| \le T, \ t_{2\nu} \ge \gamma \sum_{i=1}^{2\nu-1} t_i \right\}.$$

Theorem of Ercolani and McLaughlin [3] *For all $\nu \in \mathbb{N}$, there exist $T > 0$, $\gamma > 0$, and a neighborhood A of 1 such that the following asymptotic expansion holds for $t \in U(T,\gamma,\nu)$ and $x \in A$ as $n, N \to \infty$ with the ratio $n/N = x$ fixed:*

$$\log \frac{Z_{n,N}(t)}{Z_{n,N}(0)} \sim \sum_{g \ge 0} e_g(x,t) N^{2-2g}$$

The coefficient functions e_g are analytic for $x, t \in A \times U(T,\gamma,\nu)$, and have analytic continuations to a complex neighborhood of $t = 0$. The meaning of "$\sim$" is: for fixed ν, T, γ we have

$$\left| \log \frac{Z_{n,N}(t)}{Z_{n,N}(0)} - \sum_{g=0}^{G} e_g(x,t) N^{2-2g} \right| < C_G N^{-2g},$$

where the constant C_G is independent of x and t so long as they are in the set $U(T,\gamma,\nu) \times A$. Furthermore, this expansion can be differentiated term by term:

$$\partial_{t_1}^{m_1} \dots \partial_{t_{2\nu}}^{m_{2\nu}} \log \frac{Z_{n,N}(t)}{Z_{n,N}(0)} \sim \sum_{g \ge 0} \partial_{t_1}^{m_1} \dots \partial_{t_{2\nu}}^{m_{2\nu}} e_g(x,t) N^{2-2g}.$$

Formula of B,I,P,Z '78

Brezin, Parisi, Itzykson and Zuber 1978, [2]

$$e_0(x=1,t) = \frac{1}{2}\log z + \frac{1}{24}(z-1)(z-9), \qquad (\text{case } j=4)$$
$$z = \frac{-1+\sqrt{1+48t}}{24t}$$

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$$z = \frac{1}{16}(\beta-\alpha)^2$$

α, and β are the endpoints of the support of a random matrix eigenvalue equilibrium measure.

Formulas of B,I,Z ‘80

Bessis, Itzykson and Zuber, 1980 [1]

$$e_1(x=1,t) = -\frac{1}{12}\log(2-z), \qquad \text{(case } j=4\text{)}$$
$$e_2(x=1,t) = \frac{1}{6!}\frac{(1-z)^3}{(2-z)^5}(82+21z-3z^4)$$

Formulas of E,M,P '08

Ercolani, McLaughlin and Pierce 2008, [4]. If j is even:

$$e_0(x=1,t) = \frac{1}{2}\log z + \frac{(j-2)^2}{4j(j+2)}(z-1)\left(z - \frac{3j+6}{j-2}\right)$$

$$e_1(x=1,t) = -\frac{1}{12}\log(j/2-(j/2-1)z)$$

$$e_2(x=1,t) = \frac{(z-1)}{(j/2-(j/2-1)z)^5} \times \begin{pmatrix}\text{Polyn. of deg. 4 in } z, \\ \text{coeff's depend on } j\end{pmatrix}$$

Formulas of E,P '12

Ercolani and Pierce 2012, [5]

$$
\begin{aligned}
e_0(x=1,t) =& \frac{1}{2}\log z + \frac{1}{12}\frac{(z-1)(z^2-6z-3)}{(z+1)}, \quad (\text{case } j=3) \\
e_1(x=1,t) =& \frac{-1}{24}\log\frac{3-z^2}{2} \\
e_2(x=1,t) =& \frac{1}{960}\frac{(z^2-1)^3(4z^4-93z^2-261)}{(z^2-3)^5}
\end{aligned}
$$

What's left?

My dissertation adresses the following cases:

- General formulas for odd j
- Formulas which admit multiple time parameters

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