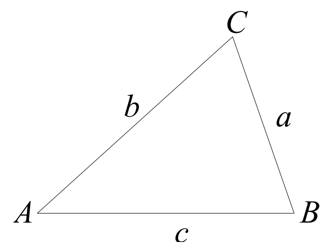
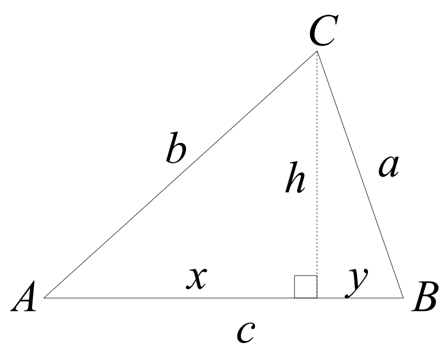
# Lesson Plans - Apr. 22

**Section 5.3: The Law of Cosines**

1. Consider the following triangle:




If we draw an altitude from vertex $C$, we get the following picture:




If we apply the Pythagorean theorem to each of the right triangles that were formed, we can see that

$$\begin{aligned} h^2 + x^2 &= b^2 \\ h^2 + y^2 &= a^2, \end{aligned}$$

and note that we already know that $x + y = c$. So, we have

$$b^2 - x^2 = a^2 - y^2,$$

but since $y = c - x$, we actually have

$$\begin{aligned} b^2 - x^2 &= a^2 - (c - x)^2 \\ b^2 - x^2 &= a^2 - [c^2 - 2cx + x^2] \\ b^2 - x^2 &= a^2 - c^2 + 2cx - x^2 \\ b^2 &= a^2 - c^2 + 2cx. \end{aligned}$$

Now, we can try to figure out what $x$ is:

$$\cos A = \frac{h}{b}, \text{ so } h = b \cos A.$$

Substituting in, we get

$$b^2 = a^2 - c^2 + 2cb \cos A \text{ or } a^2 = b^2 + c^2 - 2bc \cos A.$$

Since we can do this with any side of the triangle as the base, we in fact have the following three Laws of Cosines:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

2. What kinds of information do we need to have about a triangle in order to use the Law of Cosines? SAS or SSS.

3. What about a triangle with the information that leads to the ambiguous case of the Law of Sines? Does it help to use the Law of Cosines? Consider the following $\Delta ABC$, with $\angle A = 37°$, $b = 10$, and $a = 7$. Note that this gives us two possible triangles if we consider the Law of Sines. However, let us use the Law of Cosines:

$$7^2 = 10^2 + c^2 - 2(10)(c)\cos(37°),$$

then we get a quadratic equation:

$$49 = 100 + c^2 - 15.97c,$$

which we can see has two possible solutions for $c$. We can proceed from there to solve two possible triangles, and note that the angle $B$'s in each triangle are supplementary to each other.

4. **Examples.**

   (a) Solve triangle $\Delta XYZ$: $x = 14$, $y = 11$, $z = 5$.
   *Answer.* Let us first solve for $\angle X$:

   $$14^2 = 11^2 + 5^2 - 2(11)(5)\cos X,$$

   so we get $\cos X = 0.455$, so $X =$. Then we can use the Law of Sines to find $\angle Y$:

   $$\frac{\sin X}{14} = \frac{\sin Y}{11},$$

   So $\angle Y =$. Then, since we know $\angle X$ and $\angle Y$, we know that $\angle Z =$.

   (b) Solve triangle $\Delta RST$: $R = 43°$, $s = 35$, $t = 20$.
   *Answer.* Applying the Law of Cosines:

   $$r^2 = 35^2 + 20^2 - 2(35)(20)\cos(43°),$$

   so we get $r = 24.5$. Then we can use the Law of Sines:

   $$\frac{\sin 43°}{24.5} = \frac{\sin S}{35}.$$