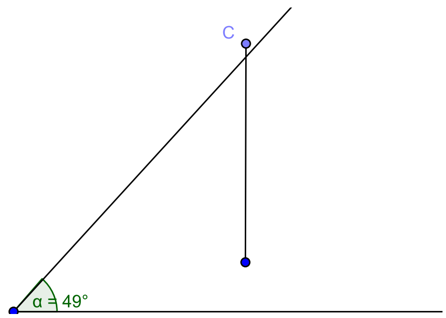
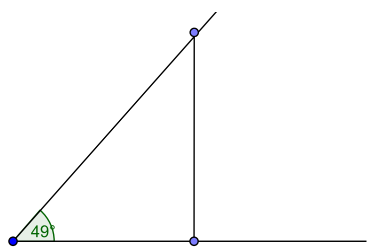
Lesson Plans - Apr. 22

Section 5.2: The Ambiguous Case of the Law of Sines (Part 2)

1. Say we are given an angle and two sides of a triangle. However, we are given $\angle A$, side a and side b - we know that $\angle A$ must be opposite side a. Let us assign numbers: $\angle A = 49°$ and $b = 10$ in. Then, we can draw the following:

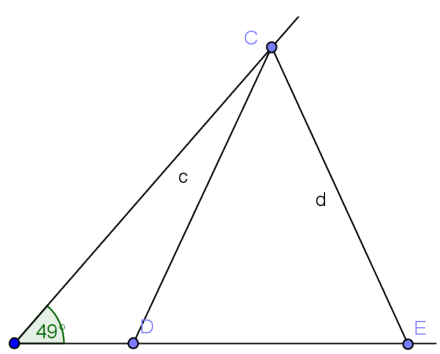


So how do we draw the third side? We know it's length, but how can we tell if it goes all they way down? The shortest length would be the segment that is perpendicular to the base of the triangle. So if we draw a right triangle, we can calculate the shortest length that the third side can be in order to complete a triangle.



We can see that this side has length $h = 10\sin(49°) \approx 7.55$. So the opposite side must be at least 7.55 inches long.

2. Assume this side is more than 7.55 inches long. Note that we can draw the side in two ways in order to make a triangle:



So, if the side opposite $\angle A$ is smaller than 7.55 inches, we cannot make a triangle at all. If the side is between 7.55 inches and 10 inches, we can make two triangles. And if the side is longer than 10 inches, we can only make one triangle. So how do we solve a triangle if we have two possibilities? How do we know which one is the correct triangle?

3. The Law of Sines actually does not help in this case. Say that we know the length of the side opposite $\angle A$ is $a = 8$ inches. There are two possible angles measures for $\angle B$:

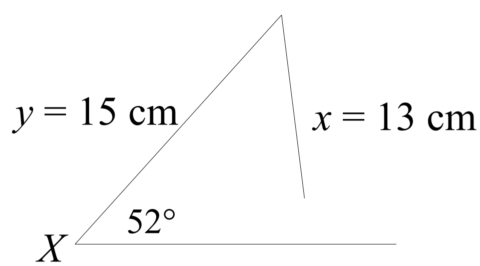
$$\frac{\sin A}{a} = \frac{\sin B}{b}, \quad \frac{\sin(49^\circ)}{8} = \frac{\sin B}{10}, \quad \sin B = \frac{\sin(49^\circ)}{8} \cdot 10 \approx 0.943.$$

But there are two possible angles that have that sine value, and they are both less than 180°. We can see this by graphing $Y_1 = \sin x$ and $Y_2 = 0.943$ in our calculators - the two curves intersect at two $x-$values which are lower than 180°. Note that if we take the inverse sine of 0.943, we will only get one possible value, the smaller value, for the angle. However, let us do that now:

$$\sin^{-1}(0.943) \approx 70.6^\circ.$$

What you should notice now, however, is that the other possible angle is supplementary to the one we just found, so the other case is when this angle is 109.4°. Once we have sorted out the two cases, we can solve the rest of each triangle. But it is important to note that you are going to solve TWO DIFFERENT triangles.

4. **Example.** Solve ΔXYZ with $X = 52^\circ$, $y = 15$ cm, and $x = 13$ cm.
Answer. First, we draw a picture: Note that we cannot draw the entire triangle yet, because we do not know if it even exists. Let us draw the parts we do know, however:



We need to know if 13 cm is long enough to reach to the base, completing the triangle. So we have to calculate the shortest length that will complete the triangle - and that is the length that is perpendicular to the base. So, then we have a right triangle, so we can calculate

$$h = 15\sin(52^\circ) \approx 11.82.$$

Since 13 is definitely larger than that we know that we can form a triangle. Also, however, we know that since $13 < 15$, we can in fact, form TWO triangles. Let us solve the acute triangle first:

$$\frac{\sin(52^\circ)}{13} = \frac{\sin Y}{15}, \text{ so } \sin Y \approx 0.909 \text{ and } Y \approx 65.4^\circ.$$

Then we know that $180^\circ - 52^\circ - 65.4^\circ = 62.6^\circ$, so $Z = 62.6^\circ$. This tells us that

$$\frac{\sin(62.6^\circ)}{z} = \frac{\sin(52^\circ)}{13}, \text{ so } z \approx 14.6.$$

We still have another triangle to solve, however, which is the following: we know that in this case $Y = 180^\circ - 65.4^\circ = 114.6^\circ$. So we can solve for Z:

$$Z = 180^\circ - 114.6^\circ - 52^\circ = 13.4^\circ.$$

And for z:

$$\frac{\sin(52^\circ)}{13} = \frac{\sin(13.4^\circ)}{z}, \text{ so } z \approx 3.82.$$