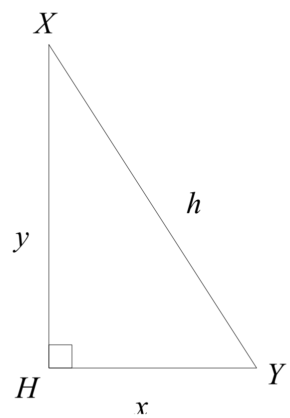
Lesson Plans - Jan. 26

Section 1.5: Fundamental Identities Trigonometric **identities** are relationships between trig functions which are ALWAYS true for all appropriate values of the variable.

1. **Reciprocal Identities**
   There are 3 trig functions that we have discussed, and 3 more which are called *reciprocal functions*. Given the following triangle,

   note the following,

   $$\begin{aligned} \sin X &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{h} \\ \cos X &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{y}{h} \\ \tan X &= \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{y}, \end{aligned}$$

   but we also have

   $$\begin{aligned} \csc X &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{h}{x} \\ \sec X &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{h}{y} \\ \cot X &= \frac{\text{adjacent}}{\text{opposite}} = \frac{y}{x}. \end{aligned}$$

   So we can see that

   $$\frac{1}{\sin X} = \frac{1}{x/h} = \frac{h}{x} = \csc X.$$

   This gives us the first three **identities**:

   $$\boxed{\begin{array}{c} \csc X = \frac{1}{\sin X} \\ \sec X = \frac{1}{\cos X} \\ \cot X = \frac{1}{\tan X} \end{array}}$$

2. **Quotient Identities**

Now, notice the following:

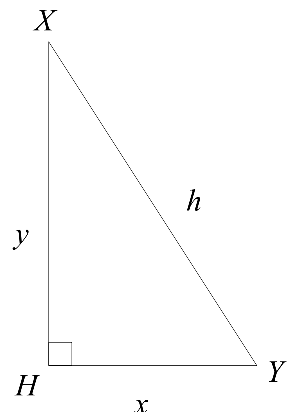
$$\frac{\sin X}{\cos X} = \frac{x/h}{y/h} = \frac{x}{y} = \tan X.$$

So we have one more identity. Remember, it does not matter what the value of X is: this is always true. In fact, if we flip all the fractions in the above equation, we get one more identity.

$$\boxed{\begin{array}{c} \frac{\sin X}{\cos X} = \tan X \\ \frac{\cos X}{\sin X} = \cot X = \frac{\csc X}{\sec X} \end{array}}$$

3. **Pythagorean Identities**

This identity is arguably one of the most important identities in trigonometry.



We know the Pythagorean Theorem:

$$x^2 + y^2 = h^2.$$

If we divide both sides of the equation by h^2, we get

$$\begin{aligned} \frac{x^2}{h^2} + \frac{y^2}{h^2} &= \frac{h^2}{h^2} \\ \left(\frac{x}{h}\right)^2 + \left(\frac{y}{h}\right)^2 &= 1 \\ (\sin X)^2 + (\cos X)^2 &= 1 \\ \sin^2 X + \cos^2 X &= 1. \end{aligned}$$

We call this identity the *Pythagorean Identity.* If, instead of dividing the Pythagorean theorem through by h^2, we pick x^2 or y^2, we will get two other EQUIVALENT forms of the identity. So we have the following:

$$\boxed{\begin{array}{c} \sin^2 X + \cos^2 X = 1 \\ \tan^2 X + 1 = \sec^2 X \\ 1 + \cot^2 X = \csc X \end{array}}$$

4. **Cofunction Identities**
The last set of identities we will talk about are *cofunction identites*. For only acute angles X, we have the following:

$$\boxed{\begin{aligned}\sin(90^\circ - X) = \cos X\\ \cos(90^\circ - X) = \sin X\end{aligned}}$$

We can see that this is true from looking at picture of ΔXYH. We can see that $\angle X$ and $\angle Y$ are complementary angles, which means that $Y = 90^\circ - X$. But,

$$\cos X = \frac{y}{h} = \sin Y = \sin(90^\circ - X).$$

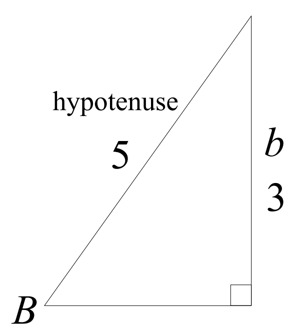
We can see that this is similarly true for the other identity.

5. Examples:

   (a) Given an acute $\angle B$, $\sin B = 3/5$. Find the EXACT values of the other five trigonometric functions of $\angle B$.
   Answer. We want to find the exact values of $\cos B$, $\tan B$, $\csc B$, $\sec B$ and $\tan B$. We can do this in two ways, the first way is to draw a right triangle. We know that

$$\sin B = \frac{\text{side opposite } \angle B}{\text{hypotenuse}} = \frac{3}{5}.$$

   So we have:



   We can use the Pythagorean identity to find the length of the leg adjacent to $\angle B$: (let us call it a)

$$5^2 = 3^2 + a^2, \text{ so } a^2 = 25 - 9 = 16, \text{ therefore, } a = 4.$$

   So, we can calculate the following trig functions of $\angle B$:

$$\begin{aligned}\cos B &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}\\ \tan B &= \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}\\ \csc B &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}\\ \sec B &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}\\ \cot B &= \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}\end{aligned}$$

OR we can use identities to calculate the trig functions we don't know. We are given that $\sin B = 3/5$. The Pythagorean identity relates the sine of an angle to its cosine. So,

$$\begin{aligned}
\sin^2 B + \cos^2 B &= 1 \\
\left(\frac{3}{5}\right)^2 + \cos^2 B &= 1 \\
\frac{9}{25} + \cos^2 B &= 1 \\
\cos^2 B &= 1 - \frac{9}{25} = \frac{16}{25} \\
\cos B &= \frac{4}{5}.
\end{aligned}$$

Then, from the quotient identity, we know that

$$\tan B = \frac{\sin B}{\cos B} = \frac{3/5}{4/5} = \frac{3}{4}.$$

Using the reciprocal identities, we can find the final three trig functions

$$\begin{aligned}
\csc B &= \frac{1}{\sin B} = \frac{1}{3/5} = \frac{5}{3} \\
\sec B &= \frac{1}{\cos B} = \frac{1}{4/5} = \frac{5}{4} \\
\cot B &= \frac{1}{\tan B} = \frac{1}{3/4} = \frac{4}{3}.
\end{aligned}$$

(b) However, say we only know that $\cos C = 0.46$. To find the rest of the trig functions, we must use identities.
Answer. Pythagorean identity:

$$\sin^2 C + \cos^2 C = 1,\ \sin^2 C + (0.46)^2 = 1,\ \sin^2 C = 1 - 0.2116 = 0.7884,\ \sin C \approx 0.888.$$

Quotient Identity:

$$\tan C = \frac{\sin C}{\cos C} = \frac{0.888}{0.46} \approx 1.93.$$

Reciprocal Identities:

$$\begin{aligned}
\csc C &= \frac{1}{\sin C} = \frac{1}{0.888} \approx 1.126 \\
\sec C &= \frac{1}{\cos C} = \frac{1}{0.46} \approx 2.174 \\
\cot C &= \frac{1}{\tan C} = \frac{1}{1.93} \approx 0.518.
\end{aligned}$$

(c) Find an acute angle X such that $\sin 38^\circ = \cos X$.
Answer. From the cofunction identites, we know that X is complementary to 38°, so $X = 90^\circ - 38^\circ = 52^\circ$.

Homework
Read pages 45-49 in the book, and do the following problems:
Section 1.5: #1, 2, 5, 8, 10, 12, 16, 17, 18