The Mathematics of Apportionment

Math 105

Section 003 - Prasad

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Apportionment

Apportionment is the science of dividing and assigning things in a proportional manner.

Basic definitions

- The population to seats ratio is called the **standard divisor**, calculated

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- The standard quota is the population of the state divided by the standard divisor:
$$\begin{aligned} \text{standard quota} &= \frac{\text{state population}}{SD} \\ &= \text{state population} \times \frac{\text{seats}}{\text{total population}} \\ &= \frac{\text{state population}}{\text{total population}} \times \text{seats}. \end{aligned}$$

Terminology

- **states**: any list of groups/people/governments etc who are being apportioned things
- **seats**: any set of identical objects that can be apportioned among the states (one seat cannot be divided up - it must be given whole)
- **populations**: the "weights" of the states - tells us the proportion of seats that is the fair share of each state

Quotas

Remember that the standard quota is

$$\text{standard quota} = \frac{\text{state population}}{SD}.$$

This is most likely NOT be a whole number. The **lower quota**, L, is the standard quota rounded down. The **upper quota**, U, is the standard quota rounded up.

$$q = 15.32, \qquad L = 15, \qquad U = 16$$

The Quota Rule

No state should get more seats that it's upper quota or fewer seats than it's lower quota.

The General Idea

All the methods have the same basic outline:

1. Pick a divisor.
2. Use this divisor to calculate each state's quota.
3. Round these quotas in some way.
4. Deal with leftover seats.

Hamilton's method

1. Find the **standard divisor**.
2. Calculate each state's **standard quota**.
3. Give each state it's **lower quota**.
4. Leftover seats: Rank the states by the "leftovers", and give each state one more seat in this order until all the seats are apportioned.

Hamilton's method

Example: If we are dividing 36 seats among 4 states with the following standard quotas:

State	A	B	C	D
Standard quota	13.21	15.65	12.12	5.02

By Hamilton's method, we would first give every state it's lower quota:

State	A	B	C	D
Lower quota	13	15	12	5

There is still one seat left. Then calculate the **residues**, which is just the standard quota minus the lower quota:

State	A	B	C	D
Standard quota	0.21	0.65	0.12	0.02

Hamilton's method

State	A	B	C	D
Standard quota	0.21	0.65	0.12	0.02

From highest to lowest residues, the states are B, A, C, D. So we hand out one seat to each state in the order on this list until we've given out all the leftover seats. In this case, there is only one leftover seat, so it goes to B. So,

State	A	B	C	D
Apportionment	13	16	12	5

Example

A parent wants to apportion 20 pieces of candy to his four kids based on the amount of time spent doing chores.

Kid	Jack	Cindy	Mike	Dana
Time spent on chores	35 min	12 min	23 min	30 min

a. Find the standard divisor.

b. Find every kid's standard quota.

c. Find the apportionment of candy using Hamilton's method.

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- However, there are a few paradoxes that Hamilton's method falls prey to.
- **Alabama paradox**: An increase in the total number of seats forces a state to lose a seat.
- **population paradox**: A state A loses a seat to state B even though the population of A grew at a higher rate than the population of B.
- **new-states paradox**: The addition of a new state with its fair share of seats can affect the apportionments of other states.

The Alabama paradox

By adding a seat to the total number of House seats, the recalculation of apportionment meant that Alabama ended up losing a seat.

	with $M = 299$		with $M = 300$	
State	**Quota**	**Apportionment**	**Quota**	**Apportionment**
Alabama	7.646	**8**	7.671	**7**
Texas	9.64	9	9.672	10
Illinois	18.64	18	18.702	19

Other's methods

	Jefferson's	**Adams'**	**Webster's**
Divisor	Find a divisor (start with SD)	Find a divisor (start with SD)	Find a divisor (start with SD)
Quota	Find the quota with the above divisor	Find the quota with the above divisor	Find the quota with the above divisor
Rounding	Apportion the **lower** quota	Apportion the **upper** quota	Round the quota above in the standard way, and apportion that
Leftovers?	Adjust D and start over	Adjust D and start over	Adjust D and start over

The Huntington-Hill Method

1. Find a divisor (start with SD).
2. Find the quota with the above divisor.
3. Round the quota with respect to the **geometric mean**, and apportion that.
4. If there are leftover seats, adjust D and start over.

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- Geometric mean of two numbers

$$G = \sqrt{a \cdot b}, \quad \text{for example, } \sqrt{3 \cdot 4} = 3.464.$$

So we would round up to 4 if the number is greater than or equal to 3.464 and down to 3 if it is less than 3.464.

Rounding geometrically

Round each quota first with respect to the arithemetic mean, then with respect to the geometric mean.

a. $q = 2.35$

b. $q = 2.45$

c. $q = 60.498$

Problem 5

The Republic of Tropicana is a small country consisting of five states (A, B, C, D, and E). The total population of Tropicana is 23.8 million. According to the Tropicana constitution, the seats in the legislature are apportioned to the states according to their populations. The following table shows each state's standard quota:

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

a. Find the number of seats in the Tropicana legislature.

b. Find the standard divisor.

c. Find the population of each state.

Problem 15

Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Hamilton's method.

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

Problem 22

A mother wishes to apportion 15 pieces of candy among her three daughters, Katie, Lilly and Jaime, based on the number of minutes each child spent studying. The only information we have is that mom will use Hamilton's method and that Katie's standard quota is 6.53.

a. Explain why it is impossible for all three daughters to end up with five pieces of candy each.

b. Explain why it is impossible for Katie to end up with nine pieces of candy.

c. Explain why it is impossible for Lilly to end up with nine pieces of candy.

Problem 27

Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Jefferson's method.

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

Problem 47

Find the apportionment of the Republic of Tropicana legislature described in Exercise 5 under Webster's method.

State	A	B	C	D	E
Standard quota	40.50	29.70	23.65	14.60	10.55

Huntington-Hill Problem 12

Round each quota using the Huntington-Hill rounding rules.

a. $q = 4.46$

b. $q = 4.48$

c. $q = 50.498$

d. $q = \sqrt{12.01}$