

Math in Modern Society

Mathematics of Power

Extra Practice

1. How many dictators can there be in a weighted voting system? Why?

There can only be one dictator in a weighted voting system. We can see that this is true by asking, what if there are two dictators? Suppose we have a weighted voting system with P_1 and P_2 both being dictators. So then, we would have $w_1 > q$ and $w_2 > q$. If we combine these two inequalities, we get

$$w_1 + w_2 > 2q,$$

which means that q is less than $\frac{w_1+w_2}{2}$. That means that q is less than half the total weight of the system, which makes it a bad quota.

2. If there is a dictator, what number player is this player?

Since there can only be one dictator, at most, it must be the player with the highest weight. Since we always write our players in descending order of weight, this means that the dictator should be P_1 .

3. If P_1 and P_3 both have veto power, does P_2 have veto power?

Yes. If a motion needs P_1 and P_3 to pass, and since $w_2 \geq w_3$, it will need P_2 to pass as well. If a motion can pass with just P_1 and P_3 , it should be able to pass with just P_1 and P_2 , meaning that P_3 would not, in fact, have veto power.

4. If P_5 and P_7 are dummies,

- (a) will P_6 be a dummy?

Yes, because $w_5 \geq w_6$, so if P_5 's weight does not make a difference to passing a motion, then P_6 's weight will not either.

- (b) will P_8 be a dummy? Yes, for a similar reason.

5. Given the weighted voting system:

$$[q : 10, w_2, 7, w_4, 5].$$

- (a) What are the possible weights for P_2 ?

The possible values of w_2 can be as high as 10 or as low as 7. So

$$7 \leq w_2 \leq 10.$$

- (b) What are the possible weights for P_4 ?

Similarly,

$$5 \leq w_4 \leq 7.$$

- (c) If each player has a different weight, what are the possible weights for P_2 ? P_4 ?

Then we have that $w_2 = 8$ or 9 , and $w_4 = 6$.

6. If P_1 has veto power, is P_2 a dummy?

Not necessarily. Consider the following weighted voting system:

$$[10 : 8, 4, 3, 2].$$

We can see that P_1 has voting power. However, if we have the coalition $\{P_1, P_2\}$, we can see that P_2 is a critical player, so it cannot be a dummy.

7. Can all players have veto power? If so, when will this occur?

Yes, if the quota is equal to the weight of the entire system, then every player needs to vote yes to pass a motion, so if any one of them votes no, the motion will not pass.

8. Given the weighted voting system:

$$[13 : w_1, 9, 2, 1, 1]$$

What are the possible weights for P_1 ?

Since the weight of the whole system should not exceed 25 (so that the quota is still more than 50% of the total weight), we can see that the max that w_1 can be is

$$25 - 9 - 2 - 1 - 1 = 12,$$

and the minimum it could be is 9, so $9 \leq w_1 \leq 12$.

9. In a weighted voting system, if there are 4 players with the following:

Player 1 has a power index of 40%,

Player 3 has a power index of 30%

(a) What is the power index of Player 2?

30%, since the power index of P_2 should be at least equal to or greater than the power index of P_3 .

(b) What is the power index of Player 4?

0%.

10. If there are 5 how many coalitions are there

(a) using the Banzhaf power index distribution?

Here we are counting the possible coalitions, of which there are $2^N - 1$. So

$$2^5 - 1 = 31.$$

(b) using the Shapley-Shubik power index distribution?

Here we are counting the possible sequential coalitions, so there are $N!$ of them. So

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$