The Mathematics of Power

Math 105

Section 003 - Prasad

September 5, 2010

Weighted Voting

- Non-weighted: One voter, one vote

Weighted Voting

- Non-weighted: One voter, one vote
- **Weighted**: One voter, n votes

Weighted Voting

- Non-weighted: One voter, one vote
- **Weighted**: One voter, n votes
- What are some of examples of weighted voting systems?

Weighted Voting

- Non-weighted: One voter, one vote
- **Weighted**: One voter, n votes
- What are some of examples of weighted voting systems?
- If all voters have 2 votes each instead of one, is this weighted? Why or why not?

Power

We want to see how much **power** each voter has under a weighted voting system, since it is likely that each voter has a different amount of power, if each voter has a different weight.

Weighted voting systems

- Only consider elections between two candidates (basically yes or no votes), called **motions**.

Weighted voting systems

- Only consider elections between two candidates (basically yes or no votes), called **motions**.
- We can express any weighted voting system as follows:

$$[q : w_1, w_2, \ldots, w_N], \text{ with } w_1 \geq w_2 \geq \ldots \geq w_N$$

where q is the **quota**, N is the number of **players**, $w_1, \ldots, w_N$ are the **weights** of players $P_1, \ldots, P_N$, respectively.

Weighted voting systems

- Only consider elections between two candidates (basically yes or no votes), called **motions**.
- We can express any weighted voting system as follows:

$$[q : w_1, w_2, \ldots, w_N], \text{ with } w_1 \geq w_2 \geq \ldots \geq w_N$$

  where q is the **quota**, N is the number of **players**, $w_1, \ldots, w_N$ are the **weights** of players $P_1, \ldots, P_N$, respectively.
- What does the following tell us?

$$[15 : 11, 9, 8, 7]$$

Weighted voting systems

$$[15 : 11, 9, 8, 7]$$

- $q = 15$ is the quota. This is how many votes are needed to pass the motion.
- There are four numbers after the quota, so there are four players.
- The weight of P_1 is $w_1 = 11$. The weight of P_2 is $w_2 = 9$. The weight of P_3 is $w_3 = 8$. The weight of P_4 is $w_4 = 7$.

Do you see any problems with this weighted voting system?
(Hint: What if P_1 and P_3 vote yes, and P_2 and P_4 vote no?)

Quota

$$[15 : 11, 9, 8, 7]$$

- We can see that this quota is too small. In this case, how many votes are there total? How many would constitute a simple majority? We want the quota to be at least as big the simple majority.

Quota

$$[15 : 11, 9, 8, 7]$$

- We can see that this quota is too small. In this case, how many votes are there total? How many would constitute a simple majority? We want the quota to be at least as big the simple majority.
- That means, if we add up all the weights (which gives us the total number of votes),

$$w_1 + w_2 + \ldots + w_N,$$

we want the quota to be at least half this number.

Quota

$$[15 : 11, 9, 8, 7]$$

- We can see that this quota is too small. In this case, how many votes are there total? How many would constitute a simple majority? We want the quota to be at least as big the simple majority.
- That means, if we add up all the weights (which gives us the total number of votes),

$$w_1 + w_2 + \ldots + w_N,$$

  we want the quota to be at least half this number.
- So,

$$q > \frac{w_1 + w_2 + \ldots + w_N}{2}.$$

Quota

$$[q : 11, 9, 8, 7]$$

- So given the weights from before, what **should** q be at least?

Quota

$$[q : 11, 9, 8, 7]$$

- So given the weights from before, what **should** q be at least?
- q should be at least 18.

Quota

$$[q : 11, 9, 8, 7]$$

- So given the weights from before, what **should** q be at least?
- q should be at least 18.
- What if the quota was set at 36? What would be the problem then?

Quota

$$[q : 11, 9, 8, 7]$$

- So given the weights from before, what **should** q be at least?
- q should be at least 18.
- What if the quota was set at 36? What would be the problem then?
- The quota should also be only as large as the sum of all the weights. In summary:

$$\frac{w_1 + w_2 + \ldots + w_N}{2} < q \leq w_1 + w_2 + \ldots + w_N.$$

Dictators, Dummies and Veto Power

- **dictator**: a player who can carry a motion without needing votes from any other players
- **dummy**: a player whose vote never really counts - this player's inclusion will never change a vote one way or another
- **veto power**: a player has veto power if a motion cannot pass without the vote of this player
- How does having veto power differ from being a dictator?

Dictators, Dummies and Veto Power

Analyze the following situations. Find all the ways each player can carry a motion. (That is, for each player, find which other players need to vote with the original player in order to pass a motion).

1. $[13 : 14, 6, 5]$
2. $[25 : 10, 8, 8, 6]$
3. $[10 : 7, 5, 4]$

Identify any dictators, dummies or players with veto power.

Coalitions

- A **coalition** is a set of players who all vote in the same way. A coalition can have as little as one player, or, if the vote is unanimous, the coalition includes all the players (this is called a **grand coalition**).

Coalitions

- A **coalition** is a set of players who all vote in the same way. A coalition can have as little as one player, or, if the vote is unanimous, the coalition includes all the players (this is called a **grand coalition**).
- A coalition that has enough votes to carry a motion is called a **winning coalition**. Otherwise, it's called a **losing coalition**. What do we call winning coalitions with only one player?

Coalitions

- A **coalition** is a set of players who all vote in the same way. A coalition can have as little as one player, or, if the vote is unanimous, the coalition includes all the players (this is called a **grand coalition**).
- A coalition that has enough votes to carry a motion is called a **winning coalition**. Otherwise, it's called a **losing coalition**. What do we call winning coalitions with only one player?
- Notation: $\{P_1, \ldots, P_m\}$ is a coalition of the first m players.

Critical players

- A player in a winning coalition is **critical** if the coalition needs that player's votes to win.

Critical players

- A player in a winning coalition is **critical** if the coalition needs that player's votes to win.
- Consider the following situation: we have a weighted voting system

  $[9 : 7, 4, 3, 2]$ and consider the following coalition $\{P_1, P_2, P_4\}$

  Which players in this coalition are critical?

Critical players

- A player in a winning coalition is **critical** if the coalition needs that player's votes to win.
- Consider the following situation: we have a weighted voting system

  $$[9 : 7, 4, 3, 2] \text{ and consider the following coalition } \{P_1, P_2, P_4\}$$

  Which players in this coalition are critical?
- The only critical player in this coalition is P_1.

Critical players

- Now consider a more general situation:

  $[q : w_1, w_2, w_3, w_4]$, with the following winning coaltion $\{P_1, P_2, P_3\}$.

  If P_1 is a critical player, what does this mean for the coalition? Can you write a mathematical statement that summarizes this?

Critical players

- Now consider a more general situation:

  $[q : w_1, w_2, w_3, w_4]$, with the following winning coaltion $\{P_1, P_2, P_3\}$.

  If P_1 is a critical player, what does this mean for the coalition? Can you write a mathematical statement that summarizes this?

- Hint, since we know this is a winning coalition we have

$$w_1 + w_2 + w_3 \geq q.$$

  What happens if you take away w_1?

Critical players

- Now consider a more general situation:

  $[q : w_1, w_2, w_3, w_4]$, with the following winning coaltion $\{P_1, P_2, P_3\}$.

  If P_1 is a critical player, what does this mean for the coalition? Can you write a mathematical statement that summarizes this?

- Hint, since we know this is a winning coalition we have

  $$w_1 + w_2 + w_3 \geq q.$$

  What happens if you take away w_1?

- So, if a player P_i is a critical player in a coalition (where the coalition has weight W), we **must** have

  $$W - w_i < q.$$

Critical players

$$[8 : 7, 5, 4]$$

Given three players, list all the possible coalitions they can form. Then, in each coalition, determine which players are critical.

Critical players

$$[8 : 7, 5, 2]$$

Possible coalitions:
$\{P_1, P_2, P_3\}$, critical players: P_1
$\{P_1, P_2\}$, critical players: P_1, P_2
$\{P_1, P_3\}$, critical players: P_1, P_3
$\{P_2, P_3\}$, losing coalition
$\{P_1\}$, losing coalition
$\{P_2\}$, losing coalition
$\{P_3\}$, losing coalition

The Banzhaf power index

- A player's power should be measured by how often the player is critical.

The Banzhaf power index

- A player's power should be measured by how often the player is critical.
- $$[8 : 7, 5, 2]$$
  Take P_1 from before. This player is critical 3 times. So we say the **critical count**, B_1, of P_1 is $B_1 = 3$.

The Banzhaf power index

- A player's power should be measured by how often the player is critical.
- $$[8 : 7, 5, 2]$$

  Take P_1 from before. This player is critical 3 times. So we say the **critical count**, B_1, of P_1 is $B_1 = 3$.
- P_2 and P_3 are each critical only once, so they each have critical counts $B_2 = 1$ and $B_3 = 1$.

The Banzhaf power index

To compute Banzhaf power

- Add up the critical counts of each of the players. In this case, we have

$$B_1 + B_2 + B_3 = 3 + 1 + 1 = 5.$$

The Banzhaf power index

To compute Banzhaf power

- Add up the critical counts of each of the players. In this case, we have

$$B_1 + B_2 + B_3 = 3 + 1 + 1 = 5.$$

- Then to calculate the Banzhaf power index of P_1, we take P_1's critical count and divide it by the total count.

$$\frac{B_1}{\text{total}} = \frac{3}{5}.$$

The Banzhaf power index

To compute Banzhaf power

- Add up the critical counts of each of the players. In this case, we have

$$B_1 + B_2 + B_3 = 3 + 1 + 1 = 5.$$

- Then to calculate the Banzhaf power index of P_1, we take P_1's critical count and divide it by the total count.

$$\frac{B_1}{\text{total}} = \frac{3}{5}.$$

- So the Banzhaf power index of P_1 (which we denote β_1) is

$$\beta_1 = \frac{3}{5} \text{ or we can express it as } \beta_1 = 60\%.$$

The Banzhaf power index

To compute Banzhaf power

- Add up the critical counts of each of the players. In this case, we have

$$B_1 + B_2 + B_3 = 3 + 1 + 1 = 5.$$

- Then to calculate the Banzhaf power index of P_1, we take P_1's critical count and divide it by the total count.

$$\frac{B_1}{\text{total}} = \frac{3}{5}.$$

- So the Banzhaf power index of P_1 (which we denote β_1) is

$$\beta_1 = \frac{3}{5} \text{ or we can express it as } \beta_1 = 60\%.$$

- What are the Banzhaf power indices of P_2 and P_3?

Banzhaf power distribution

We can collect these power indices into a **Banzhaf power distribution**, which is just an ordered list of the indices:

$$\beta_1 = 60\%, \beta_2 = 20\%, \beta_3 = 20\%.$$

Computing the number of possible coalitions

Given 4 players, how many possible coalitions do we have? 5 players? n players?

Think about it this way: for each player, you have the choice to include or exclude that player from the coalition. Derive (and justify) a mathematical expression for the number of possible coalitions given n players.

Computing the number of possible coalitions

Let us consider 4 players for now. We can create coalitions by including or excluding certain players.

P_1	P_2	P_3	P_4	coalition
Y	Y	Y	Y	$\{P_1, P_2, P_3, P_4\}$
N	Y	Y	Y	$\{P_2, P_3, P_4\}$
Y	N	Y	Y	$\{P_1, P_3, P_4\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Computing the number of possible coalitions

We can see from this we have $2^4 = 16$ possible coalitions, but one is the empty coalition (that is, coalition where we have excluded everybody). So, in fact, we have $2^4 - 1 = 15$ possible coalitions.

In general, given n players, we have $2^n - 1$ possible coalitions.

Note that $\{P_1, P_2, P_4\}$ is the same coalition as $\{P_2, P_4, P_1\}$. We only care about the **combination** of players in the coalition, not the order in which they are listed.

Example

Consider the following weighted voting system:

$$[31 : 29, 28, 3].$$

Compute the Banzhaf power distribution for this system.

Shapley-Shubik Power

A **sequential coalition** is a coalition in which the order of the players matter. For sequential coalitions, we will always list **every** player.

$$\langle P_1, P_2, P_3 \rangle$$
$$\langle P_1, P_3, P_2 \rangle$$
$$\langle P_2, P_1, P_3 \rangle$$
$$\langle P_2, P_3, P_1 \rangle$$
$$\langle P_3, P_1, P_2 \rangle$$
$$\langle P_3, P_2, P_1 \rangle$$

Sequential coalitions

How many sequential coalitions are possible if you have four players? (Hint: what does this remind you of from chapter 1?)

Sequential coalitions

How many sequential coalitions are possible if you have four players? (Hint: what does this remind you of from chapter 1?)

$$4! = 4 \times 3 \times 2 \times 1 = 24 \text{ possible sequential coalitions}$$

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.
- There is only ever **one** pivotal player in any sequential coalition.

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.
- There is only ever **one** pivotal player in any sequential coalition.
- Given the following weighted voting system:

$$[10 : 5, 4, 3, 3, 1]$$

Find the pivotal player in each of the following coalitions:

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.
- There is only ever **one** pivotal player in any sequential coalition.
- Given the following weighted voting system:

$$[10 : 5, 4, 3, 3, 1]$$

  Find the pivotal player in each of the following coalitions:

  1. $\langle P_1, P_2, P_3, P_4, P_5 \rangle$

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.
- There is only ever **one** pivotal player in any sequential coalition.
- Given the following weighted voting system:

$$[10 : 5, 4, 3, 3, 1]$$

Find the pivotal player in each of the following coalitions:

1. $\langle P_1, P_2, P_3, P_4, P_5 \rangle$
2. $\langle P_5, P_2, P_1, P_4, P_3 \rangle$

Pivotal players

- The **pivotal player** in a sequential coalition is the player whose votes turn a losing coalition into a winning one.
- There is only ever **one** pivotal player in any sequential coalition.
- Given the following weighted voting system:

$$[10 : 5, 4, 3, 3, 1]$$

Find the pivotal player in each of the following coalitions:

1. $\langle P_1, P_2, P_3, P_4, P_5 \rangle$
2. $\langle P_5, P_2, P_1, P_4, P_3 \rangle$
3. $\langle P_2, P_5, P_4, P_1, P_3 \rangle$

The Shapley-Shubik Power Index

The Shapley-Shubik power index of a player P_i is calculated by

$$SS_i = \frac{\text{number of times } P_i \text{ is a pivotal player}}{\text{total number of times all the players are pivotal}}.$$

The Shapley-Shubik Power Index

Consider the following weighted voting system

$$[6 : 4, 3, 2].$$

Calculate the Shapley-Shubik power index (SS_i) for each player.

Example 1

Consider the weighted voting system: $[q : 8, 4, 1]$.

a. What are the possible values of q?
b. Which of these values of q result in a dictator? Who is it?
c. Which values of q result in exactly one player with veto power? Again, who is it?
d. Which values of q result in more than one player with veto power? Explain.
e. Which values of q result in one or more dummies?

Example 1

Consider the weighted voting system: $[q : 8, 4, 1]$.

a. Possible values of q: $7 \leq q \leq 13$.

b. P_1 is a dictator for $q = 7$ and $q = 8$.

c. Only P_1 has veto power for $q = 7$ and $q = 8$.

d. For $10 \leq q \leq 12$, P_1 needs P_2 to pass a motion, and for $q = 13$, all the players have veto power.

e. For $10 \leq q \leq 12$, P_1 needs only P_2 to pass a motion, so P_3 is a dummy. For $q = 7, 8$, P_1 is a dictator, so P_2 and P_3 are dummies.

Example 2

We defined a **dummy** as a player that is never critical. Explain why each of the following is true:

a. If P is a dummy, then any winning coalition that contains P would also be a winning coalition without P.

b. P is a dummy if and only if the Banzhaf Power Index of P is 0.

c. P is a dummy if and only if the Shapley-Shubik Power Index of P is 0.

Example 3

Verify that the weighted voting systems

$$[12 : 7, 4, 3, 2] \text{ and } [24 : 14, 8, 6, 4]$$

result in exactly the same Banzhaf power distribution. Then verify that they result in the same Shapley-Shubik power distribution.

Extra Practice Questions

1. How many dictators can there be in a weighted voting system? Why?
2. If there is a dictator, what number player is this player?
3. If P_1 and P_3 both have veto power, does P_2 have veto power?
4. If P_5 and P_7 are dummies,
   a. will P_6 be a dummy?
   b. will P_8 be a dummy?

Extra Practice Questions

5. Given the weighted voting system:

$$[q : 10, w_2, 7w_4, 5].$$

   a. What are the possible weights for P_2?
   b. What are the possible weights for P_4?
   c. If each player has a different weight, what are the possible weights for P_2? P_4?

6. If P_1 has veto power, is P_2 a dummy?
7. Can all players have veto power? If so, when will this occur?

Extra Practice Questions

8. Given the weighted voting system:

$$[13 : w_1, 9, 2, 1, 1]$$

   What are the possible weights for P_1?

9. In a weighted voting system, if there are 4 players with the following:

   Player 1 has a power index of 40%,
   Player 3 has a power index of 30%

   a. What is the power index of Player 2?
   b. What is the power index of Player 4?

10. If there are 5 how many coalitions are there
    a. using the Banzhaf power index distribution?
    b. using the Shapley-Shubik power index distribution?