Math 105 - Math in Modern Society

Chapter 2 Solutions

Sept. 13, 2010

10. (a) We know that all three players have veto power if the quota equals the weight of the whole system. So we know that $q = 15$ will definitely work, but we want the **smallest** value of q where they all have veto power. Note that for $q \leq 12$, P_1 and P_2 can carry a motion on their own, so P_3 does not have veto power. However, for $q = 13$, we need all three players to vote yes in order for the motion to pass. So, $q = 13$ is the least value of q for which all three players have veto power.

    (b) We saw above that P_3 has veto power for $q \geq 13$, so we know that $q \leq 12$ for P_3 to not have veto power. However, q needs to be at least 8 to be a valid quota, and we can see that P_1 cannot carry a motion singlehandedly. But, if $q = 8$ or 9 or 10, P_1 only needs P_3 to carry a motion - and can ignore P_2. So P_2 has veto power, and P_3 does not, for $q = 11$ and $q = 12$. So $q = 11$ is the smallest value where this is true.

    (c) Again, let us start with all the possible values of q: $8 \leq q \leq 15$. P_3 has veto power for $13 \leq q \leq 15$, and P_3 is a critical player in the coalition $\{P_1, P_3\}$, for $8 \leq q \leq 10$. So, P_3 is can only possibly be a dummy for $q = 11$ or $q = 12$. Suppose $q = 11$. Let us look at all the coalitions that include P_3:

    $$\{P_3\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_1, P_2, P_3\}.$$

    Only $\{P_1, P_2, P_3\}$ is a winning coalition, and P_3 is not a critical player in that coalition. So, $q = 11$ is the smallest value of q for which P_3 is a dummy.

47. If we write out this voting system, we have

    $$[q : 8x, 4x, 2x, x].$$

    Since q is a simple majority, we want to be slightly bigger than 50% of the total weight of the system. Let us split this up into cases:
    Case 1: Let x be even. We can see that the quantity $\frac{15x}{2}$ will be a whole number (any even number divided by 2 is still a whole number), so the quota will have to be that quantity plus 1, so

    $$q = \frac{15x}{2} + 1.$$

    Since we want to prove that P_1 will be a dictator, we want to show that the inequality

    $$\frac{15x}{2} + 1 \leq 8x$$

    is always true. If we solve this inequality, we get:

    $$\begin{aligned} 2\left(\frac{15x}{2} + 1\right) &\leq 2(8x) \\ 15x + 2 &\leq 16x \\ 2 &\leq 16x - 15x \\ 2 &\leq x. \end{aligned}$$

    This tells us that for all even numbers x that are greater than or equal to 2, the inequality is true, which means that the weight of P_1 is equal to or larger than the quota. So, for all even numbers x greater than or equal to 2, P_1 is a dictator.

Case 2: Let x be odd. We can see that the quantity $\frac{15x}{2}$ will **not** be a whole number, so the quota will have to be that quantity plus $\frac{1}{2}$, so

$$q = \frac{15x}{2} + \frac{1}{2}.$$

Since we want to prove that P_1 will be a dictator, we want to show that the inequality

$$\frac{15x}{2} + \frac{1}{2} \leq 8x$$

is always true. If we solve this inequality, we get:

$$\begin{aligned} 2\left(\frac{15x}{2} + \frac{1}{2}\right) &\leq 2(8x) \\ 15x + 1 &\leq 16x \\ 1 &\leq 16x - 15x \\ 1 &\leq x. \end{aligned}$$

This tells us that for all odd numbers x that are greater than or equal to 1, the inequality is true, which means that the weight of P_1 is equal to or larger than the quota. So, for all odd numbers x greater than or equal to 2, P_1 is a dictator.

So, putting the two statements together, we have that for all whole numbers x greater than 0, P_1 is a dictator. Since those are the only values of x that we care about (why don't we care about 0?), we can say that P_1 is always a dictator.

48. (a) We already know what all the winning coalitions are, but we need to figure out who the critical players are in each one. Remember that we can tell if a player is critical in a winning coalition if the coalition becomes a losing coalition without them. We have enough information to figure this out, even though we don't know any numbers. Consider the first winning coalition: $\{P_1, P_2, P_3\}$. If P_1 is critical, then the coalition without P_1 is a losing coalition. If we take P_1 out of this coalition, we get $\{P_2, P_3\}$ left, which we know is losing, since it's not on the list of winning coalitions. Therefore, P_1 must be critical. We need to do this for every player in every coalition, so we get

$$\begin{array}{c} \{\underline{P_1}, \underline{P_2}, \underline{P_3}\} \\ \{\underline{P_1}, \underline{P_2}, \underline{P_4}\} \\ \{\underline{P_1}, \underline{P_3}, \underline{P_4}\} \\ \{\underline{P_1}, P_2, P_3, P_4\}. \end{array}$$

So that we have Banzhaf counts: $B_1 = 4$, $B_2 = 2$, $B_1 = 2$, $B_4 = 2$. So, we have Banzhaf power distribution

$$\beta_1 = 40\%, \quad \beta_2 = 20\%, \quad \beta_3 = 20\%, \quad \beta_4 = 20\%.$$

(b) Now we are considering sequential coalitions. Consider one such coalition: $\langle P_1, P_2, P_3, P_4\rangle$. We know a player is pivotal if they tip the coalition from losing to winning. So let us proceed one player at a time: $\langle P_1, \ldots$ is not the pivotal player in this coalition. If P_1 were pivotal, then the coalition $\{P_1\}$ would be a winning one. $\langle P_1, P_2 \ldots$, P_2 is not pivotal, because if it were, $\{P_1, P_2\}$ would be a winning coalition, which we know it is not. However, for $\langle P_1, P_2, P_3, \ldots$, we can see that P_3 is pivotal in this coalition, since $\{P_1, P_2, P_3\}$ is a winning coalition. So we can write $\langle P_1, P_2, \underline{P_3}, P_4\rangle$. We need to do this for every sequential coalition (yes, all 24 of them!).

$$\begin{array}{cccc}
\langle P_1, P_2, \underline{P_3}, P_4\rangle & \langle P_3, P_1, \underline{P_2}, P_4\rangle & \langle P_2, P_1, \underline{P_3}, P_4\rangle & \langle P_4, P_1, \underline{P_2}, P_3\rangle \\
\langle P_1, P_2, \underline{P_4}, P_3\rangle & \langle P_3, P_1, \underline{P_4}, P_2\rangle & \langle P_2, P_1, \underline{P_4}, P_3\rangle & \langle P_4, P_1, \underline{P_3}, P_2\rangle \\
\langle P_1, P_3, \underline{P_2}, P_4\rangle & \langle P_3, P_2, \underline{P_1}, P_4\rangle & \langle P_2, P_3, \underline{P_1}, P_4\rangle & \langle P_4, P_2, \underline{P_1}, P_3\rangle \\
\langle P_1, P_3, \underline{P_4}, P_2\rangle & \langle P_3, P_2, P_4, \underline{P_1}\rangle & \langle P_2, P_3, P_4, \underline{P_1}\rangle & \langle P_4, P_2, P_3, \underline{P_1}\rangle \\
\langle P_1, P_4, \underline{P_2}, P_3\rangle & \langle P_3, P_4, \underline{P_1}, P_2\rangle & \langle P_2, P_4, \underline{P_1}, P_3\rangle & \langle P_4, P_3, \underline{P_1}, P_2\rangle \\
\langle P_1, P_4, \underline{P_3}, P_2\rangle & \langle P_3, P_4, P_2, \underline{P_1}\rangle & \langle P_2, P_4, P_3, \underline{P_1}\rangle & \langle P_4, P_3, P_2, \underline{P_1}\rangle
\end{array}$$

Then we have the following Shapley-Shubik counts: $SS_1 = 12$, $SS_2 = 4$, $SS_3 = 4$, $SS_4 = 4$. The Shapley-Shubik power distribution is

$$\sigma_1 = \frac{1}{2}, \quad \sigma_2 = \frac{1}{6}, \quad \sigma_3 = \frac{1}{6}, \quad \sigma_4 = \frac{1}{6}.$$

52. (a) Suppose there is a winning coalition which does not include that player (lets call that player P_1). This means that there is at least one way for the other players to pass a motion without P_1. This violates the definition of P_1 having veto power.

(b) Suppose our player, P_1, is not a critical player in the grand coalition. This means that the coalition which includes all the other players but not P_1 is a winning coalition, which, from part (a), violates the definition of P_1 having veto power.

54. (a) P_5 is the only dummy. Consider all the winning coalitions which include P_5. They have coalition weights 31 or 41. Without P_5, these weights would be 30 and 40, which means they are still winning coalitions. So P_5 is not critical in any of the winning coalitions that it is part of. Therefore, it is a dummy.

(b) Since P_5 is a dummy, it has power index 0% under both power indices. Since the other players all have equal weight, they should all have equal power indices. So we have

$$\beta_1 = 25\%, \quad \beta_2 = 25\%, \quad \beta_3 = 25\%, \quad \beta_4 = 25\%, \quad \beta_5 = 0\%,$$

and

$$\sigma_1 = 25\%, \quad \sigma_2 = 25\%, \quad \sigma_3 = 25\%, \quad \sigma_4 = 25\%, \quad \sigma_5 = 0\%.$$

(c) P_5 is not a dummy for $q = 21, 31, 41$.

(d) The values of w that make P_5 a dummy are $w = 1, 2, 3$.

56. (a) Since the quota is 9, and the quota needs to be higher than half the total weight, the most that the total weight can be is 17. So the highest that w can be is 9. Since the weight of P_2 is 5, w has to be at least that big, so $5 \leq w \leq 9$.

(b) P_1 is a dictator for all values of $w \geq 9$, but since there is only value of w that fits, we have $w = 9$.

(c) Since the coalition $\{P_2, P_3, P_4\}$ has weight 8, it is not a winning coalition, they always need P_1 to pass a motion. So, for all the values of w in part (a), P_1 has veto power.

(d) If $w = 9$, then the rest of the players are dummies. For $w = 8$, the coalition $\{P_1, P_4\}$ is a winning coalition, with P_4 critical. So, if even P_3 is not a dummy, there are no dummies. For $w = 7$, we can see that P_4 is a dummy. For $w = 6$, again we have a winning coalition in which P_4 is critical, so there are no dummies. For $w = 5$, both P_3 and P_4 are dummies.