Math in Modern Society
Notes on Mathematics of Voting
Part 1

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1 Basics

1.1 Ballots

In order to make things simpler, we like our ballots to have the following properties:

- In a **preference ballot**, voters are asked to rank the candidates in order of preference. For example, if they are given a choice of Alice, Bob and Cara, a voter's ballot might look like:

Ballot
Bob
Cara
Alice

- In a **linear ballot**, ties are not allowed, so voters must rank the candidates in some hierarchy. They must decided between any two candidates.

A **preference schedule** groups together identical ballots, so instead of looking at many individual ballots, we need only consider a manageable number of categories. Suppose we have three candidates: Alice, Bob and Cara. There are six different ways to rank these candidates:

Ballot Types

Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice

So, we can sort each of our ballots into one of 6 categories. Note there will not necessarily be every possible ranking of candidates represented. A preference schedule would therefore look like this:

Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

1.2 Facts about Voting

- Voter preferences are **transitive**.
- A voter's relative preferences are not affected by the **elimination** of one or more of the candidates.

2 Fairness Criteria

2.1 The Majority Criterion

If a candidate has a **majority** (more than half) of the first-place votes, then that candidate is the **winner** of the election. We call this candidate the **majority candidate**.

2.2 The Condorcet Criterion

If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election. We call this candidate the **Condorcet candidate**.

2.3 An Example

Consider our previous example: we have an election with three candidates. Suppose that a preference ballot is used. Suppose that there were 30 votes, with the following preference schedule:

Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

1. Who got the most first place votes? Alice
2. Who is the majority candidate? none
3. Who is the Condorcet candidate? Cara
   Alice vs. Bob: 16 prefer Alice, 14 prefer Bob
   Alice vs. Cara: 16 prefer Cara, 14 prefer Alice
   Bob vs. Cara: 18 prefer Cara, 12 prefer Bob
4. Who would be the winner if Alice dropped out of the election? Cara
5. If Bob dropped out? Cara
6. If Cara? Alice

2.4 Further Questions

Consider an election with four candidates.

1. How many possible ways are there to rank these candidates? Can you come up with a way to calculate this without listing all the possible rankings? If so, use this to calculate how many ways there are to rank 8 candidates.

2. Create an example using these four candidates where the plurality candidate is different from the Condorcet candidate.

3. In your example, is there a majority candidate? If this is not already the case, is there a way to tweak your numbers to get a majority candidate that is not the Condorcet candidate? If not, explain.

4. Choose a winner for the election and defend your choice (from the example you created in #2). Why did you pick one criterion over the other?