The Mathematics of Voting

Math 105

Section 003 - Prasad

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- **linear ballot:** ties are not allowed
- **preference schedule:** group together identical ballots

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- There are six possible rankings:

Ballot 1	Ballot 2	Ballot 3	Ballot 4	Ballot 5	Ballot 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice

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Bob	Cara	Alice	Cara	Alice	Bob
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- So, we can sort each of our ballots into one of 6 categories. Note there will not necessarily be every possible ranking of candidates represented.

Voter Preferences

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- A voter's relative preferences are not affected by the **elimination** of one or more of the candidates.

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- **Methods of counting** are ways in which we count votes, given a preference schedule.
- **Fairness criteria** are how we judge voting methods.
- Using a fairness criterion, we can determine who we *think* the winner of an election should be. Then, we use a method of counting to actually count the votes. We would like the candidate with the most votes to be the candidate the fairness criterion tells us it should be.

The Majority Criterion

If a candidate has a **majority** (more than half) of the first-place votes, then that candidate is the **winner** of the election. We call this candidate the **majority candidate**.

The Condorcet Criterion

If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election. We call this candidate the **Condorcet candidate**.

Alice, Bob and Cara

Consider our previous example. We have an election with three candidates. Suppose that a preference ballot is used. Suppose that there were 30 votes, with the following preference schedule:

Type 1	Type 2	Type 3	Type 4	Type 5	Type 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

- Who got the most first place votes (called the **plurality candidate**)?
- Who is the majority candidate?
- Who is the Condorcet candidate?

Results

- Most first-place votes: Alice
- Majority candidate: none
- Condorcet candidate: Cara
  Alice vs. Bob: 16 prefer Alice, 14 prefer Bob
  Alice vs. Cara: 16 prefer Cara, 14 prefer Alice
  Bob vs. Cara: 18 prefer Cara, 12 prefer Bob
- Who would be the winner if Alice dropped out of the election? If Bob dropped out? If Cara?

Pass out notes now!

Further Questions

Consider an election with four candidates.

1. How many possible ways are there to rank these candidates? Can you come up with a way to calculate this without listing all the possible rankings? If so, use this to calculate how many ways there are to rank 8 candidates.
2. Create an example using these four candidates where the plurality candidate is different from the Condorcet candidate.
3. In your example, is there a majority candidate? If this is not already the case, is there a way to tweak your numbers to get a majority candidate that is not the Condorcet candidate? If not, explain.
4. Choose a winner for the election and defend your choice (from the example you created in #2). Why did you pick one criterion over the other?

The Monotonicity Criterion

If candidate X is the winner of an election and, in a reelection, the only changes in the ballots are changes that favor X (and only X), then X should remain a winner of the election.

The Independence-of-Irrelevant Alternatives Criterion

If candidate X is a winner of an election and in a recount one of the non-winning candidates is removed from the ballots, then X should still be a winner of the election.

The Plurality Method

- The candidate with the most first-place votes wins.
- We call this candidate the **plurality** candidate.

This is the most common method of voting - this is how we run local and state-wide elections, for example.

The Borda Count Method

Borda Count: this method assigns values to each place of the ranking.

First choice	3 points
Second choice	2 points
Third choice	1 point

What are the Borda points for each candidate?

Ballot 1	Ballot 2	Ballot 3	Ballot 4	Ballot 5	Ballot 6
Alice Bob Cara	Alice Cara Bob	Bob Alice Cara	Bob Cara Alice	Cara Alice Bob	Cara Bob Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

Results

Ballot 1	Ballot 2	Ballot 3	Ballot 4	Ballot 5	Ballot 6
Alice (3) Bob (2) Cara (1)	Alice (3) Cara (2) Bob (1)	Bob (3) Alice (2) Cara (1)	Bob (3) Cara (2) Alice (1)	Cara (3) Alice (2) Bob (1)	Cara (3) Bob (2) Alice (1)
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

Borda points					
Alice (18) Bob (12) Cara (6)	Alice (21) Cara (14) Bob (7)	Bob (3) Alice (2) Cara (1)	Bob (15) Cara (10) Alice (5)	Cara (9) Alice (6) Bob (3)	Cara (24) Bob (16) Alice (8)

Total Borda points:

Alice	Bob	Cara
60	56	64

So Cara is also the **Borda winner**.

The Plurality-with-Elimination Method

This method is good for getting a majority candidate out of the plurality method if you have three or more candidates.

1. Count the number of first-place votes. If there is a majority candidate, this candidate is the winner. If not, eliminate the candidate with the fewest *first-place* votes.
2. Recount the first place votes now that you have eliminated a candidate. If there is now a majority candidate, this candidate is the winner.
3. Continue to eliminate and recount until there is a majority candidate.

Alice, Bob and Cara, again

Remember our preference schedule from before:

Ballot 1	Ballot 2	Ballot 3	Ballot 4	Ballot 5	Ballot 6
Alice	Alice	Bob	Bob	Cara	Cara
Bob	Cara	Alice	Cara	Alice	Bob
Cara	Bob	Cara	Alice	Bob	Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

Who is the winner under this method?

Results

The first place votes are

Alice: 13
Cara: 11
Bob: 6

So we eliminate Bob from the ballots.

Ballot 1	Ballot 2	Ballot 3	Ballot 4	Ballot 5	Ballot 6
Alice	Alice	~~Bob~~	~~Bob~~	Cara	Cara
~~Bob~~	Cara	Alice	Cara	Alice	~~Bob~~
Cara	~~Bob~~	Cara	Alice	~~Bob~~	Alice
6 voters	7 voters	1 voter	5 voters	3 voters	8 voters

And we can see that Alice now has 14 first-place votes, while Cara has 16 first-place votes. So Cara is the winner.

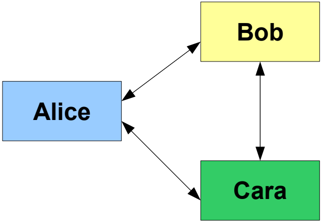
The Pairwise Comparisons Method

For an election with three or more candidates, this method evaluates each pair of candidates, finds a winner, and assigns that candidate one point. (If there is a tie, they both get half a point.)

Suppose we have Alice, Bob and Cara again. How many head-to-head (or **pairwise**) comparisons do we need to make?

Pairwise comparisons of N candidates

With Alice, Bob and Cara, we have three candidates. An easy way to think about it is as a diagram.



So we have three comparisons to make. But what if there are more than three candidates?

Pairwise comparisons of N candidates

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- Consider the first candidate in your list, C_1. You need to compare C_1 to the other four candidates:

$$C_1 \text{ vs. } C_2 \quad C_1 \text{ vs. } C_3 \quad C_1 \text{ vs. } C_4 \quad C_1 \text{ vs. } C_5$$

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- Now the second candidate, C_2, had already been compared to C_1, so we need only compare C_2 to three of the other candidates:

$$C_2 \text{ vs. } C_3 \quad C_2 \text{ vs. } C_4 \quad C_2 \text{ vs. } C_5$$

Pairwise comparisons of N candidates

- Then you've already compared C_3 to C_1 and C_2. So you should only have to compare C_3 to the remaining two candidates:

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- Finally, you only need to do one last comparison:

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- So we have to make

$$4 + 3 + 2 + 1 = 10 \text{ comparisons.}$$

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- The first candidate needs to be compared to all $N - 1$ other candidates.

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- The second needs to be compared to all but the first, so $N - 1 - 1 = N - 2$ candidates.

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What if you have some general N number of candidates? You do not want to make a list, if N is a large number.

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- The second needs to be compared to all but the first, so $N-1-1=N-2$ candidates.
- If we continue, we can see that the total number of comparisons you need to make are:

$$(N-1)+(N-2)+(N-3)+\ldots+4+3+2+1.$$

Large Consecutive Sums

Let $M = N - 1$. How can we figure out what the number

$$M + (M - 1) + (M - 2) + \ldots + 4 + 3 + 2 + 1$$

is?

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- So $2S = M(M+1)$. So we get

$$S = \frac{M(M+1)}{2}$$

Large Consecutive Sums

1. Calculate the following sum:

$$1 + 2 + 3 + 4 + \ldots + 170 + 171 + 172$$

2. How many pairwise comparisons do we need to make if we have 10 candidates?

Back to Alice, Bob and Cara

Since there are three candidates, there are (as we showed before), for $M = 3 - 1 = 2$

$$\frac{M(M+1)}{2} = \frac{2(2+1)}{2} = 3 \text{ comparisons.}$$

- Let us consider each of these head-to-head comparisons:

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- Alice vs. Bob: Alice gets the point.

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- Alice vs. Bob: Alice gets the point.
- Alice vs. Cara: Cara gets the point.
- Bob vs. Cara: Cara gets the point.
- So the winner is **Cara**.

Condorcet criterion vs. Pairwise Comparisions Method

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- Therefore, it is possible for an election under the pairwise comparison method to have winner, but not a Condorcet winner. But the Condorcet winner always wins the method of pairwise comparisons.
- So, we say that the method of pairwise comparisons **satisfies** the Condorcet criterion.

Fairness violations of the Plurality Method

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- Explain why the plurality method satisfies the majority criterion.
- Give an example with four candidates that shows how the plurality method violates the Condorcet criterion. (Hint: you want to create an example where the Condorcet candidate is different from the plurality candidate.)

Fairness violations of the Borda Count Method

Determine the Borda winner of the election with the following preference schedule:

Number of voters	6	2	3
First choice	A	B	C
Second choice	B	C	D
Third choice	C	D	B
Fourth choice	D	A	A

Who is the majority candidate? What about the Condorcet candidate?

Fairness violations of the Plurality-with-Elimination Method

This method violates the following fairness criteria:

- Monotonicity criterion
- Condorcet criterion

There is an example of this in your book that you should re-read carefully.

Fairness violations of Pairwise Comparisons

This method violates the independence-of-irrelevant-alternatives criterion (IIA for short). There is an example of this in your homework.

Arrow's Impossibility Theorem

It is mathematically **impossible** for a democratic voting system to satisfy all fairness criteria, in an election with three or more candidates.

Extra Practice Handout

Explain why, under the plurality method, if you have a majority candidate (who is the plurality candidate), this candidate must also be a Condorcet candidate.

Further Questions

Suppose you had the following fairness criterion:

If a majority of the voters have candidate X ranked last, then candidate X should not be a winner of the election.

1. Give an example to show that the plurality method violates this criterion.
2. Give an example to show that the plurality-with-elimination method violates this criterion.
3. Explain why the method of pairwise comparisons satisfies this criterion.
4. Explain why the Borda count method satisfies this criterion.