Math 105 - Math in Modern Society

Chapter 1 Solutions

Sept. 1, 2010

14. There are 150 votes to count, and after counting 120, we have the following totals:

A	B	C	D
26	18	42	34

(a) We want to find the **least** number of votes that B needs for a certain win. We can see that if B gets all 30 remaining votes, then B wins. However, is there a smaller number of votes that will guarantee victory? We see that if B gets 25 votes, then B will have a total of 43 votes, and that will be enough to beat the current leader, C, so long as C does not get any more votes. But there are still 5 votes unaccounted for, and if C gets any of those votes, C will win. So we need to give B a vote for every vote C can get out of those remaining 5. If we do this, we see that we need to give B a total of **28 votes**. Then we would, **at worst**, have the following distribution:

A	B	C	D
26	46	44	34

.

(b) We can do a similar thing to what we did above. Given our original distribution:

A	B	C	D
26	18	42	34

we have to give D 8 votes in order for D to match C. Then both C and D have 42 votes, and we need to apportion out the remaining 22 votes. D needs to get more than half of these votes (in case the rest of them go to C), so we need to give D another 12 votes, meaning D should get a total of **20 votes**.

20. (a) We can calculate the following Borda counts:

$$\begin{aligned} A &= 4(4)+1(3)+9(2)+8(4)+5(1)=74, \\ B &= 4(3)+1(4)+9(1)+8(2)+5(2)=51, \\ C &= 4(1)+1(1)+9(4)+8(1)+5(4)=69, \\ D &= 4(2)+1(2)+9(3)+8(3)+5(3)=76. \end{aligned}$$

We see that the Borda winner is D.

(b) The majority of first-place votes went to candidate C, who got 14 votes out of 27. So C is the majority candidate. However, under this method, D is the winner of the election, so this election violates the majority criterion.

(c) If we look at the following head-to-head comparisons, we can see that C is also the Condorcet candidate:

$\underline{A}$ vs B
A vs $\underline{C}$
A vs $\underline{D}$
B vs $\underline{C}$
B vs $\underline{D}$
$\underline{C}$ vs D

But again, D was the winner under this method, so this election violates the Condorcet criterion.

24. (a) The way for a candidate to earn as many points as possible is for that candidate to be everyone's first choice candidate. Then, for every ballot, this candidate will earn 5 points. Since there are 20 voters, there will be 20 ballots cast, so the most points a candidate can get is

$$5 \text{ points per ballot } \times 20 \text{ ballots } = 100 \text{ points.}$$

(b) The way for a candidate to get the fewest number of points possible is for that candidate to be everyone's last-place candidate. Then, for every ballot, this candidate will earn only 1 point (but note that they will earn a point! They will not earn 0 points!). Since there are 20 voters, there will be 20 ballots cast, so the most points a candidate can get is

$$1 \text{ point per ballot } \times 20 \text{ ballots } = 20 \text{ points.}$$

(c) Each voter ranks the five candidate, and hands out 5 points to their first place candidate, 4 points to their second place candidate, 3 points to their third place candidate, 2 points to their fourth place candidate and 1 point to their last place candidate. So each voter hands out

$$5 + 4 + 3 + 2 + 1 = 15 \text{ points.}$$

(d) The total number of points, then, is

$$15 \text{ points per voter } \times 20 \text{ voters } = 300 \text{ points total.}$$

(e) Since the points have to add up to 300, we can calculate the number of points candidate E gets as follows:

$$300 - 69 - 70 - 64 - 48 = 49 \text{ points.}$$

30. (a) Remember that in plurality-with-elimination, we have to go through the elimination rounds.

Round 1:	A	B	C	D
	12	1	14	0
Round 2:	A	B	C	
	12	1	14	
Round 3:	A	C		
	13	14		

So C is the winner under plurality-with-elimination.

(b) C is the majority candidate, so C will always have the most first-place votes, no matter how many votes get apportioned to anyone else through the rounds. No other candidate can get enough votes to add up to more than fifty percent of the votes.

(c) If there is a majority candidate, there is no way any other candidate can gain enough votes to beat the majority candidate. In the final round, the majority candidate already has the most votes (more than fifty percent).

56. (a) In a round-robin tournament, every player plays every other player exactly once. If there are 21 players, then the first player has to play 20 other players, then the second player has to play another 19 players (not including the first player), etcetera. So there are

$$1 + 2 + 3 + \ldots + 19 + 20 = \frac{20(21)}{2} = 210 \text{ matches.}$$

(b) If they can play 6 matches per hour, it takes

$$\frac{20 \text{ matches}}{6 \text{ matches per hour}} = 35 \text{ hours.}$$

Then they can only play for 12 hours a day, so

$$\frac{35 \text{ hours}}{12 \text{ hours per day}} \approx 3 \text{ days.}$$

61. If there are only two candidates, both plurality and plurality-with-elimination work the same way, and the winner is whoever has the most first-place votes, and that winner is also automatically the majority candidate. Note the pairwise-comparison method has only one pairwise comparison to make, so whoever has more first-place votes is automatically the winner (and has the majority of the votes). For the Borda count method, consider the following example:

Percentage of voters	51%	49%
First place	A	B
Second place	B	A

Two-thirds of the points go to first-place votes, and one-third goes to second-place votes. Since candidate A will get $\frac{2}{3} \times 51\% + \frac{1}{3} \times 49\%$ of the Borda points, and candidate B will get $\frac{1}{3} \times 51\% + \frac{2}{3} \times 49\%$ of the points. In other words,

$$\text{Candidate A: } \frac{2}{3} \times \frac{51}{100} + \frac{1}{3} \times \frac{49}{100} = \frac{151}{300} \text{ of the points,}$$

while candidate B only gets $\frac{149}{300}$ of the points. So the majority candidate, candidate A, wins under the Borda count method as well.

62. Here is one example of a preference schedule that satisfies all the criteria:

Number of voters	8	4	3	2
First place	A	B	B	D
Second place	C	D	C	C
Third place	B	C	D	B
Fourth place	D	A	A	A

You should verify this on your own.