The Mathematics of Money

Math 105

Section 003 - Prasad

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New Groups for Finance Unit

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- This works the other way if we borrow money - if we borrow X amount of money now, we have have to pay back more than that in the future.
- Finance is, basically, the art of calculating a future value.

Percents

Def: A **percent** is just a fraction with the denominator of 100.

So, if we have a fraction like $\frac{4}{15}$, we are trying to figure out what the numerator of the equivalent fraction would be if you wanted the denominator to be 100:

$$\frac{4}{15} = \frac{x}{100} \to 4(100) = 15x \to x = \frac{400}{15} \approx 26.67$$

So, we are saying that

$$\frac{4}{15} \approx \frac{26.67}{100},$$

but an easier way to say this is that $\frac{4}{15}$ is 26.67%.

Percents and decimals

Another thing to remember is that the decimal 0.53 means

$$\frac{53}{100}.$$

This is **why** 0.53 corresponds to 53%. However, we are usually talking about finding percents **of** quantities, and these quantities are not necessarily 100.

Percentages

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2. What is 17% of 2175.24?

Problem 81 on page 396

How much should a retailer mark up her goods so that when she has a 25% off sale, the resulting prices will still reflect a 50% markup (on her cost)?

Problem 83 on page 397

You have a coupon worth $x\%$ off any item (including sale items) in a store. The particular items you want is on sale for $y\%$ off the marked price of $\$P$.

1. Give an expression for the price of the item assuming that you first go $y\%$ off sale price and then had the additional $x\%$ taken off using your coupon.
2. Give an expression for the price of the item assuming you got the discounts the other way around. Compare and contrast your answers, and explain.

(Assume x and y are both positive integers smaller than 100).

Simple Interest

We calculate **simple interest** on an amount of money (called the **principal**) by adding a certain percentage (called the **interest rate**) of the principal to itself.

For example, if I invest \$3000 at a simple interest rate of 3.7% per year, how much money will I have in 15 years? 10? Fill out a chart like the one I started to figure this out:

Time in years	Interest added	Total money
0	0	\$3000
1	$\$3000(0.037) = \$111.1$	3111.1

Can you simplify this into a formula in terms of the number of years t?

Simple interest

We get the formula:

$$\text{Total money (future value)} = \$3000(1 + 0.037t)$$

We can see that this relationship will be similar for any principal P and interest rate r, so we can generalize to

$$F = P(1 + rt),$$

where F stands for the future value.

Simple interest - recap

The formula for simple interest is

$$F = P(1 + rt),$$

where F stands for the future value, P stands for the amount of money originally invested or borrowed, r is the interest rate and t is the number of times interest is added.

So, if we invest $400 at an annual simple interest rate of 2.7%, how much money will we have in 15 years?

What if that interest rate was not annual but quarterly? How many times would interest be added in 15 years? What would the future value of the money be?

Compound interest

Suppose you put \$500 in the bank at 4.1% annual interest. The following chart is started for you. Fill in the rest.

Time in years	Interest Calculation	Total Money
0	0	\$500
1	\$500(0.041) = \$20.50	\$520.50
2	\$520.50(0.041) = \$21.34	\$541.84
3	\$541.84(0.041) = \$22.22	\$564.06
4	\$564.06(0.041) = \$23.13	\$587.18
5	\$587.18(0.041) = \$24.07	\$611.25
6	\$611.25(0.041) = \$25.06	\$636.31
7	\$636.31(0.041) = \$26.09	\$662.40
8	\$662.40(0.041) = \$27.16	\$689.56

Compound interest calculation

After future value after one year is

$$F = \$500 + \$500(0.041) = \$500(1.041) = \$520.50$$

After two years, it is

$$F = \$520.50 + \$520.50(0.041) = \$520.50(1.041) = \$541.84$$

We can rewrite that as

$$[\$500(1.041)](1.041) = \$500\left[(1.041)^2\right] = \$541.84$$

Notice that if we accrue interest twice (once after each year), it's the same as multiplying 1.041 to the principal twice. Actually, after t years, the amount of money you will have is

$$F = \$500(1.041)^t$$

Compound interest formula

So, we can say that if we invest $\$P$ at an interest rate of r and compound interest t times, the future value, F is

$$F = P(1 + r)^t.$$

What if we have \$500 that we invest at an annual interest rate of 4.1%, but interest is compounded *quarterly* (this means, four times a year)? How much money will we have in 5 years?

Continuous compounding

We can see that the more often we compound interest, the more money we have. So, ideally, we would like interest to be compounded **continuously**. To do this, we use the formula:

$$F = Pe^{rt},$$

where F is the future value of $\$P$ compounded continuously for t years at an annual interest rate of r.

Annual percentage yield (APY)

The APY of an investment is what percentage of an investment the annual profit is.

In other words, if you start at the beginning of the year with \$100 and at the end of one year you have \$102.50, you made a profit of \$2.50. So the APY of this investment is

$$\text{APY} = \frac{\text{Profit}}{\text{Money at the beginning of the year}} = \frac{2.50}{100} = 0.025, \text{ or } 2.5\%.$$

Comparing Investments

Suppose you have the following three investment schemes:

- 5.2% APR compounded annually
- 5.1% APR compounded monthly
- 4.8% APR compounded continuously

How could you figure out which one is better? APY is a good way to do this. APY will tell you what percent of your original investment you will make in profit after one year. Which of these investments has the best rate of return?

Geometric Sequences

Definitions:

- A **sequence** is list of numbers which follows a pattern:

$$1, 2, 3, 4, 5, \dots \text{ or } 5, 10, 15, 20, \dots \text{ or } 2, 4, 8, 16, 32, \dots$$

- A **geometric sequence** is one in which each term is equal to some multiple of the term before it:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

- The number you multiply is called the **common ratio**, usually denoted c.

Geometric Sum

We can sum up the first N terms of a geometric sequence with the formula (where P is the **intial term**):

$$P + cP + c^2P + c^3P + \ldots + c^{N-1}P = P\left(\frac{c^N - 1}{c - 1}\right)$$

For example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$

Since the initial term $P = 1$, the common ratio is $\frac{1}{2}$ and there are $N = 7$ terms that we are adding up, we have:

$$1\left[\frac{(\frac{1}{2})^7 - 1}{\frac{1}{2} - 1}\right] = 1\left[\frac{\frac{1}{128} - 1}{-\frac{1}{2}}\right] = 1\left[\frac{-\frac{127}{128}}{-\frac{1}{2}}\right] = 1 \cdot \frac{127}{64} = \frac{127}{64}$$

Annuities

A **fixed annuity** is a sequence of equal payments made over regular periods of time (for example, monthly or quarterly, or even annually). There are two types of fixed annuities:

- A **deferred annuity** is one in which regular payments are made first, and the “reward” (the lump sum of money) comes first.
  - For example, paying money into a college trust fund or saving up to put a down payment on a house or car.
- An **installment loan** (or **immediate annuity**) is one in which you get something now (like a car), and pay it off in regular payments afterward.

Deferred annuities

Suppose you want to save up some money to put a down payment on a car. You save \$100 a month in a savings account that pays 3.0% annual interest compounded monthly. Assume you deposit money at the start of each month, and that interest is credited at the end of each month. How much money do you have at the end of two years? (Hint: make a table).

Month	Deposit	Interest earned	Prev. bal.	Total
1	\$100	$\$100(0.0025) = \$0.25$	0	\$100.25
2	\$100	$\$100(0.0025)$ $+\$100.25(0.0025) = \$0.50$	\$100.25	\$200.75
3	\$100		\$200.75	

Deferred annuities

Notice that we are basically doing the following calculation:

$$\$100(1.0025) + \$100(1.0025)^2 + \$100(1.0025)^3 + \ldots + \$100(1.0025)^{24}$$

This is a geometric sum! We can see that the initial term is $P = 100(1.0025)$, the common ratio is $c = 1.0025$, and there are $N = 24$ terms that we are adding up. So, we get:

$$F = \$100(1.0025)\left[\frac{(1.0025)^2 4 - 1}{1.0025 - 1}\right] = \$2476.46$$

Deferred annuities

Or more generally, if P is the value of each payment, T is the total number of payments, and c is the common ratio (one plus the periodic interest rate), we can express this as

$$F = Pc\left[\frac{c^T - 1}{c - 1}\right]$$

Deferred annuities

Since $1.0025 = c = 1 + p$, where p is the periodic interest rate, and $100(1.0025)$ is the last payment plus the interest it earns in one month, we can write

$$F = L\left[\frac{(1+p)^T - 1}{p}\right],$$

where $L = P(1+p)$, the value of the final payment plus the interest it earns over the final period of time. This is the formula found in the book, but it might be more helpful to understand the formula as a geometric sum.

Installment loans

The most important distinction between an installment loan and a fixed deferred annuity is that

- an installment loan has a **present** value that we compute by adding the present value of each payment,
- whereas a fixed deferred annuity has a **future** value that we compute by adding the future value of each payment.

(Page 383 in the book)

Installment loans - Example

Suppose you buy a car, and after the down payment, you have $\$12,000$ left to pay. You take out a car loan for 36 months at 5.99% APR to pay this off. What are your monthly payments?

First let us consider the following: say you pay $100 in the first month (this usually will happen at the end of the month). Your loan has already generated some interest, so that $100 is paying off both some of the loan, and some of the interest.

Installment loans - Example

To figure out how much of the \$100 pays off the principal and how much pays off the interest, we can consider the \$100 to the future value of a payment:

$$F = P(1+p)^t,$$

where p is the periodic interest rate $(0.0599/12 = 0.004992)$ and t is the time in months - in this case, $t = 1$. We want to find P:

$$\$100 = P(1+0.004992)^1 \rightarrow P = \frac{\$100}{1.004992} = \$99.50.$$

Installment loans - Example

For the second month, we put in another \$100, but this time, more of this money is paying off interest than in the first month. So the amount of principal that gets paid off is:

$$\$100 = P(1 + 0.004992)^2 \to P = \$99.01.$$

For the third month:

$$\$100 = P(1 + 0.004992)^3 \to P = \$98.52.$$

Installment loans - Example

In fact, note that we have the following:

$$\frac{\$100}{1+0.004992}, \frac{\$100}{(1+0.004992)^2}, \frac{\$100}{(1+0.004992)^3},$$

where p is the periodic interest rate and F is the fixed monthly payment.

This is a geometric sequence, and to see how much money we've added up, we need to use the geometric sum formula. Note that in this case, the common ratio is $c = \frac{1}{1.004992}$, and the intial term is $\frac{100}{1.004992}$. Let $N = 36$ (that is, 36 months).

Installment loans - Example

So, applying our geometric sum formula

$$\frac{\$100}{1.004992}\left[\frac{\left(\frac{1}{1.004992}\right)^{36}-1}{1.004992}\right]=\$3287.57.$$

We can see that this is nowhere near what we need. Moreover, it's less than what we pay if there was no interest at all! This number is the amount of the principal we have paid off. The rest of the money ($\$3600 - \$3287.57 = \$312.43$) went to paying off interest.

Installment loans - Generalization

In fact, the payments on the principal of the loan will always follow the sequence

$$\frac{F}{1+p}, \frac{F}{(1+p)^2}, \frac{F}{(1+p)^3}, \frac{F}{(1+p)^4}, \ldots, \frac{F}{(1+p)^N},$$

for a total of N payments. We can calculate the sum:

$$P = \left(\frac{F}{1+p}\right)\left[\frac{\left(\frac{1}{1+p}\right)^N - 1}{\frac{1}{1+p} - 1}\right] = \frac{F}{p}\left[1 - \left(\frac{1}{1+p}\right)^N\right],$$

where F is the fixed monthly payment, p is the periodic interest rate, and N is the number of payments made.

Installment loans - back to the example

So, back our example of the car loan: You have $12,000 left to pay. You take out a car loan for 36 months at 5.99% APR to pay this off. What are your monthly payments?

$$12000 = \frac{F}{0.004992}\left[1 - \left(\frac{1}{1.004992}\right)^{36}\right],$$

so we can solve for F:

$$F = \$365.01.$$