Remarks on Homework 4

October 24, 2006

As I mentioned to a number of you in my office, the problem of showing that the set of elements $\{e_I = e_{i_1} \cdots e_{i_k}\}$ for $i_1 < \cdots < i_k$ and $k \leq \dim V$, is a linearly independent set in the Clifford algebra, $\mathrm{Cliff}(V, S)$, is not completely trivial. However, when the dimension, $\dim V = n$, is even there is a simple argument for this that uses representation theory. The group algebra of the Clifford group $CL(n)$ is the set of complex linear combinations,

$$\sum_{\pm, I} f_I(\pm)(\pm e_I).$$

In the irreducible representation of $CL(n)$ of dimension $2^{\frac{n}{2}}$ the action of the group algebra can be identified with the action of the Clifford algebra since $\pm e_I$ is represented by $\pm\gamma_I$. However, as Simon shows, (see theorem III.1.5) the group algebra acts as the full matrix algebra in an irreducible representation of the group. Thus in this representation, ρ, on the complex vector space W, the Clifford algebra, $\mathrm{Cliff}(V, S)$, is mapped homomorphically onto the full matrix algebra, $\mathrm{Hom}(W)$, which has dimension $2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} = 2^n$. This shows that the dimension of $\mathrm{Cliff}(V, S)$ is not less than 2^n which shows that the spanning set $\{e_I\}$ must be a basis.

Now suppose that $\dim V = n + 1$ with n even and that $\{e_1, e_2, \ldots, e_{n+1}\}$ is an orthonormal basis for V, that is,

$$S(e_i, e_j) = \delta_{i,j}.$$

We define a representation of $\mathrm{Cliff}(V, S)$ on $W \otimes C^2$ by mapping,

$$e_j \to \rho(e_j) = \gamma_j \in \mathrm{Hom}(W) \text{ for } j = 1, \ldots, n$$

and

$$e_{n+1} \to \Gamma \otimes \sigma,$$

where $\Gamma = i^{\frac{n(n-1)}{2}} \gamma_1 \cdots \gamma_n$ and $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The image of $\mathrm{Cliff}(V, S)$ in this representation evidently consists of all linear transformations of the form $A \otimes e$ or $A \otimes \sigma$ where $A \in \mathrm{Hom}(W)$ and e is the 2×2 identity matrix. The dimension of the image is thus $2 \cdot 2^n = 2^{n+1}$. This shows that the dimension of $\mathrm{Cliff}(V, S)$ is at least 2^{n+1} in this case as well.