Homework 6: Math 559

November 11, 2006

1. The group $\mathrm{SO}(n)$ acts transitively on S^{n-1} by sending $(g, x) \in \mathrm{SO}(n) \times S^{n-1}$ to gx. Let $e_1 \in S^{n-1}$ denote the first standard basis vector in R^n and define $\pi : \mathrm{SO}(n) \to S^{n-1}$ by $\pi(g) = ge_1$. Show that all the fibers, $\pi^{-1}(x)$ of this map are homeomorphic to $\mathrm{SO}(n-1)$. Show also that $\mathrm{SO}(n)$ is a locally trivial fiber bundle over S^{n-1} .

Hint: S^{n-1} is covered by the two open sets $U = S^{n-1} - \{e_1\}$ and $V = S^{n-1} - \{-e_1\}$. It is enough to show that $\pi^{-1}(U)$ is homeomorphic to $U \times \mathrm{SO}(n-1)$, $\pi^{-1}(V)$ is homeomorphic to $V \times \mathrm{SO}(n-1)$ so that in each case the diagram,

$$\begin{array}{ccc} X \times \mathrm{SO}(n-1) & \rightarrow & \pi^{-1}(X) \\ \downarrow & & \downarrow \\ X & \rightarrow & X \end{array}$$

commutes. Here $X = U, V$. The top horizontal arrow is the desired homeomorphism, the left vertical arrow is the projection on the first factor, the right vertical arrow is π and the bottom horizontal arrow is the identity. In this case it is possible to choose the homeomorphisms so that the "change of trivializations" on $U \cap V \times \mathrm{SO}(n-1)$ has the form,

$$(p, g) \to (p, \phi(p)g),$$

where $\phi : U \cap V \to \mathrm{SO}(n-1)$ is a smooth map and $\phi(p)g$ is just the product in $\mathrm{SO}(n-1)$. Fiber bundles with groups as fibers and trivializations of this sort are called principal bundles.

For the final part of the hint I want to suggest how to find the homeomorphism from $U \times \mathrm{SO}(n-1)$ to $\mathrm{SO}(n)$. For $p \in U$ define a rotation $R(p)$ so that $\pi R(p) = p$ (how about the rotation through the "short arc" of the great circle through e_1 and p that rotates e_1 into p). Then define the homeomorphism by,

$$(p, g) \to R(p) \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \in \mathrm{SO}(n).$$

Find a formula for $R(p)$ and you will have no trouble verifying that this map is smooth.

This result is true for the projection $\pi : G \to G/H$ where G is a Lie group and H is a closed subgroup. If you want to give yourself an idea just how helpful having the geometric model S^{n-1} for the quotient above was, try to prove that $G \to G/H$ is a locally trivial fiber bundle. What must be true for the foliation of G associated with H to get this to work? If you find a simple proof let me know and we'll publish.

2. Now let $\pi : E \to B$ be a locally trivial fiber bundle with fiber homeomorphic to the path connected Lie group F. We suppose that E is path connected Lie group and as always π is a surjective continuous map. As above, a trivialization of E over an open set $U \subset B$ is a fiber preserving homeomorphism,

$$\phi_U : \pi^{-1}(U) \to U \times F.$$

We suppose that B is covered by open sets associated with trivializations so that for any two such trivializations, ϕ_U, ϕ_V there is a continuous map,

$$\phi_{UV} : U \cap V \to F,$$

so that,

$$\phi_U \phi_V^{-1}(p, g) = (p, \phi_{UV}(p) \cdot g) \text{ for } (p, g) \in U \cap V \times F.$$

The situation is intended to mimic what is true for $E = \mathrm{S0}(n)$, $B = S^{n-1}$ and $F = \mathrm{SO}(n-1)$ above. Show that paths and homotopies can be lifted from B to E so that there is an exact sequence of homotopy groups,

$$\pi_1(F, e) \to \pi_1(E, p) \to \pi_1(B, q) \to 0,$$

where e is the identity in F and $\pi(p) = q$. As I mentioned in class this is the beginning of a long exact sequence for fibrations that involves the higher homotopy groups.