

Problem set 2

1st September 2006

1. Use the fact that the sum of the squares of the dimensions of the irreducible representations equals the order of the group and the fact that the dimension of an irreducible must divide the order of the group to find the dimensions of the irreducible representations of the permutation groups S_2 , S_3 , and S_4 . It is helpful to observe that you already know two one dimensional representations, the trivial representation and the signature representation. Determine the character tables for S_2 and S_3 . To find the character table for S_3 observe that the conjugacy classes of S_3 are $\{e\}$, $\{(12), (23), (13)\}$, and $\{(123), (213)\}$. Use orthogonality of the characters to find the character of the third irreducible for S_3 (don't forget the "weights" for the conjugacy classes and use the fact that the trace of the identity is equal to the dimension of the representation). In the book the conjugacy classes are abbreviated: $e^1, (12)^3, (123)^2$ – identified by an element of the orbit and its size.

2. The group S_3 acts on \mathbf{C}^3 by linear extension of $U(\sigma)e_j = e_{\sigma(j)}$. Decompose this representation into irreducibles. Can you find an explicit model for the irreducible 2 dimensional representation of S_3 . Hint: the vector $[1, 1, 1]^T$ spans an invariant subspace.

3. The group S_3 acts on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ by permuting the factors. That is, it acts by linear extension of the action: $U(\sigma)v_1 \otimes v_2 \otimes v_3 = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes v_{\sigma^{-1}(3)}$. Determine the irreducibles and the multiplicities with which they appear in this representation. Hint: determine the character of this representation.

4. The group S_4 acts on \mathbf{C}^4 by linear extension of $U(\sigma)e_j = e_{\sigma(j)}$. Show directly that S_4 acts irreducibly on the orthogonal complement of the vector $[1, 1, 1, 1]^T$, and calculate the character of this representation. Hint: it is enough to show that there are no one dimensional invariant subspaces.