559 problem set 1

August 22, 2006

1. Suppose that G is a finite group and that G acts *freely* on a Hausdorff topological space X. This means that each map $\tau_g : X \to X$ is continuous and that the isotropy group, I_x, for any $x \in X$ is trivial. That is $I_x = \{e\}$ where e is the identity in G. Show that the orbit space X/G is Hausdorff and the projection,

$$\pi : X \to X/G,$$

is a covering space map.

2. Show that $\mathbf{Z}/2\mathbf{Z}$ is the only non trivial finite group that can act freely on an even dimensional sphere. This action is given by sending 0 to the identity and 1 to the antipodal map $x \to -x$. Hint: Any homeomorphism of an even dimensional sphere that does not have a fixed point is homotopic to the antipodal map and hence has degree -1. Recall that the degree of a composition of two homomeomorphisms is the product of the degrees.

3. Find a free action of $\mathbf{Z}/n\mathbf{Z}$ on any odd dimensional sphere. Here n is any positive integer. Hint: think of S^{2n+1} as a subset of $\mathbf{C^{n+1}}$.

In general the problem of determining which finite groups can act freely on a given topological space is unsolved. For spheres it is known that free group actions exist for any finite group with the property that all abelian subgroups are cyclic and every element of order 2 is central.