Final 559

May 1, 2007

1. Determine the Young symmetrizer $e(T)$ for the tableau with (1,2) in the first row and (3) in the second row. Confirm by a calculation that $e(T)^2$ is a multiple of $e(T)$. Do the same for the tableau with rows, (1,2), (3), (4).

2. The group SU(n) acts on the four fold tensor product $(\mathbf{C}^n)^{\otimes 4}$. For $n = 3$ determine the reduction of this representation into irreducibles (which highest weight representations occur) and find their dimensions and multiplicities (you can use the fact that the dimensional of a representation of S_4 associated with a Young frame is equal to the number of standard tableau associated to the frame). Do the same for $n = 4$.

2. Theorem IX.9.1 in Simon, translates the Weyl character formula into determinant form for SU(n). In class we proved the formula

$$\prod_{j=1}^{r}(x_1^{\ell_j} + x_2^{\ell_j} + \cdots + x_n^{\ell_j}) = \sum_{\mathcal{F}} \chi_{\mathcal{F}}^U(D(x))\chi_{\mathcal{F}}^S(C).$$

where $\chi_{\mathcal{F}}^U$ is the character of the representation of SU(n) associated with the frame $\mathcal{F}$, $\chi_{\mathcal{F}}^S$ is the character of the representation of the symmetric group associated with the frame $\mathcal{F}$, $D(x)$ is the diagonal matrix with entries $x_1, x_2, \ldots, x_n$, and C is a conjugacy class with cycles of size $\ell_1, \ell_2, \ldots, \ell_r$. The sum is over all m frames with fewer than n rows. Specialize this formula to $n = m = 3$ (i.e., SU(3) and S_3) and use it and Theorem IX.9.1 to figure out the character table for S_3. In general the specialization to $n = m$ in this formula is known as the Frobenius character formula (for S_n). Hint: in order to isolate the characters of the symmetric group first clear the denominators in the Weyl character formula. Then note that the "highest weight" contribution to the determinant in the numerator for $\chi_F^U(D(x))$(thought of as a polynomial in x_j) is different for each $\mathcal{F}$. This makes it possible to isolate the character $\chi_{\mathcal{F}}^S$ as the coefficient of a suitable monomial on the left. "Highest weight" for the monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ gives more weight to x_1 than x_2 more weight to x_3 than x_4 and so on.