Comments on the Final

May 14, 2007

2. In the second problem the Weyl dimension formula was used to calculate the dimensions of the irreducibles of SU(n) into which the tensor product decomposes. There is a subtlety in the application of the dimension formula that confused me, so I thought I would share it with you. The problem can be seen for SU(3) so I will restrict to this case. The basic representation (the identity representation) has three weights $\omega_1, \omega_2, \omega_3$. Of course, for SU(3) these weights are not independent but satisfy the relation,

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

Note that this relation is not compatible with the inner product $\langle \omega_i, \omega_j \rangle = \delta_{ij}$. For this inner product the square of the length of $\omega_1 + \omega_2 + \omega_3$ is $3 \neq 0$. Let D_j denote the diagonal 3×3 matrix with a 1 in the jj slot and 0's elsewhere. The real Cartan subalgebra, $h_{\mathbf{R}}$, for SU(3) is the vector space of linear combinations,

$$a = a_1 D_1 + a_2 D_2 + a_3 D_3$$

with $a_1 + a_2 + a_3 = 0$. If

$$b = b_1 D_1 + b_2 D_2 + b_3 D_3$$

is a second such vector, then the Killing form on $h_{\mathbf{R}}$ is a multiple of,

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The linear functionals defined by $\omega_i(D_j) = \delta_{i,j}$ restrict, of course, to linear functionals on $h_{\mathbf{R}}$. To find the inner products of these linear functionals we first find vectors $w_i \in h_{\mathbf{R}}$so that,

$$\omega_i(x) = \langle w_i, x \rangle \text{ for x} \in h_{\mathbf{R}}.$$

Then the inner product on the dual $h^*_{\mathbf{R}}$is given by,

$$\langle \omega_i, \omega_j \rangle = \langle w_i, w_j \rangle.$$

Without difficulty one may check that,

$w_1 = [2/3, -1/3, -1/3]^{\tau}, \quad w_2 = [-1/3, 2/3, -1/3]^{\tau}, \quad w_3 = [-1/3, -1/3, 2/3]^{\tau}.$

This gives the inner products,

$$\langle \omega_i, \omega_i \rangle = \frac{2}{3}, \text{ for } i = 1, 2, 3$$

$$\langle \omega_i, \omega_j \rangle = -\frac{1}{3}, \text{ for } i \neq j$$

This system of inner products is, of course, consistent with the sum of the ω_j being 0 (note that: $w_1 + w_2 + w_3 = 0$). Lets use this to compute the Weyl dimension formula for U_λ where $\lambda = 2\omega_1 + 2\omega_2$ (just to pick an example at random). First we calculate that $\delta = 3\omega_1 + 2\omega_2 + \omega_3 = 2\omega_1 + \omega_2$. Note that the reduction in the last step in justified in the dual of the Cartan subalgebra for SU(3) but not in the dual of the Cartan subalgebra for U(3) where ω_3 is an independent weight. We then calculate the Weyl denominator,

$$\langle \omega_1 - \omega_2, \delta \rangle \langle \omega_1 - \omega_3, \delta \rangle \langle \omega_2 - \omega_3, \delta \rangle = 1 \cdot 2 \cdot 1 = 2,$$

where, for example $\langle \omega_1, \delta \rangle = \frac{4}{3} - \frac{1}{3} = 1$ and $\langle \omega_2, \delta \rangle = -\frac{2}{3} + \frac{2}{3} = 0$. The numerator is,

$$\langle \omega_1 - \omega_2, \delta + \lambda \rangle \langle \omega_1 - \omega_3, \delta + \lambda \rangle \langle \omega_2 - \omega_3, \delta + \lambda \rangle = 1 \cdot 4 \cdot 3 = 12,$$

where, for instance $\langle \omega_1, \delta + \lambda \rangle = 5/3$. Thus,

$$\dim U_\lambda = \frac{12}{2} = 6.$$

As you all know (since this is the way you all did the dimension calculation) the calculations using the inner product $\langle \omega_i, \omega_j \rangle = \delta_{i,j}$ look completely different yet produce identical factors in the numerator and denominator. This inner product is appropriate for the dual of the Cartan subalgebra of U(3). There $\delta = 3\omega_1 + 2\omega_2 + \omega_3$ does not reduce to $2\omega_1 + \omega_2$ and yet this reduction seems to produce the correct results in the dimension formula. I don't really understand why this is so.

Of course, U(n) is not semi-simple so it is perhaps not wise to carry over results for semi-simple Lie groups to this case without some further analysis. For n=3 the dimensions and multiplicities were,

$$15 + 3 \times 15 + 2 \times 6 + 3 \times 3 = 81$$

and for n=4,

$$35 + 3 \times 45 + 2 \times 20 + 3 \times 15 + 1 = 256$$

3. In the Weyl character formula for the tensor product of 3 copies of the basic representation of SU(3)one finds that after clearing denominators the right

hand side is a linear combination of the characters $\chi^S_{(3)}, \chi^S_{(2,1)}$, and $\chi^S_{(1,1,1)}$, of the symmetric group with polynomials in x_1, x_2 and and x_3 as coefficients. It is easy to check that $\chi^S_{(3)}$ is the coefficient of $x_1^5 x_2$, $\chi^S_{(2,1)}$ is the coefficient of $x_1^4 x_2^2$ and $\chi^S_{(1,1,1)}$ is the coefficient of $x_1^2 x_2$ on the right hand side. Isolating the coefficients of these powers on the left hand side gives the characters of the symmetric group S_3. This sort of isolation works quite generally and produces the Frobenius formula for the characters of S_n. One does not need to evaluate at special points.