New trends in nonlinear analysis: asymptotic analysis in random tilings, graphs, and probability

Preparatory discussions and a few exercises

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This short manuscript provides some background material that will be useful during discussions which will take place on Saturday, pertaining to some current research on asymptotic behavior in probability and combinatorics. In the first section you will find a definition of the Aztec Diamond of order n, and a few questions related to *random tilings*.

In the second section you will find an example of an important combinatorial question: how do you count *non-intersecting paths*?

The two sections are closely related! In fact, to understand the behavior of random tilings of the Aztec Diamond when the size, n, grows to ∞, we use an amazing correspondence between tilings and non-intersecting paths. That connection, if all goes well, will be explained in Tucson.

1 Random Tilings of the Aztec Diamond of order n

The *Aztec diamond of order* n refers to a region in the plane: the union of all unit squares with corners on the integer lattice, lying within the interior of the region given by $|x|+|y| \leq n+1$. See the following links

`http://www.math.wisc.edu/~propp/tiling/www/aztec-definition.html`

`http://www.math.wisc.edu/~propp/tiling/www/intro.html`

`http://www.math.wisc.edu/~propp/tiling/www/mdblum/arctic.html`

We will refer to this region as $AD(n)$.

Tiling $AD(n)$: This refers to taking a bunch of "dominoes" - rectangles of dimension 2 units by 1 unit - and laying them on top of $AD(n)$ so that

1. Every square of $AD(n)$ is covered,
2. None of the dominoes overlap,
3. None of the dominoes cover squares outside $AD(n)$.

There is a formula for the number of possible tilings of $AD(n)$:

$$\#\text{ of tilings of } AD(n) \;=\; 2^{\frac{n(n+1)}{2}} \tag{1}$$

Now let's think like a probabilist. So there is a huge "box" which contains within it every possible tiling of $AD(n)$. Each tiling is equally likely, so the probability of picking any particular tiling is $2^{-n(n+1)/2}$.

Question: What is the probability that a tiling picked at random has the property that *there is a vertical tile stuck in one of the top squares of* $AD(n)$*?*

Hint: if you start tiling $AD(n)$ *by placing a vertical domino at the top, how many subsequent dominoes are forced? How many such tilings of* $AD(n)$ *are there?*

Challenge question: Find a proof of formula (1).

2 The combinatorics of non-intersecting paths

2.1 A fundamental example

Consider the infinite planar graph whose vertices, V, are the following subset of the integer lattice. $V = \{(m, n) : m \text{ and } n \text{ are integers, } m + n \text{ is even}\}$. The edges are oriented (we say *directed*) by specifying them as vectors between a pair of vertices in V as follows: From every vertex $(m, n) \in V$ exactly two directed edges emanate, one going from (m, n) to $(m+1, n+1)$ and one going from (m, n) to $(m+1, n-1)$. This graph is an example of an infinite acyclic digraph. For background on such graphs see

`http://en.wikipedia.org/wiki/Directed_acyclic_graph.`

We may think of any (finite) path on this digraph as a 1-dimensional walk, where the $x-$coordinate is a time variable and the walk takes place in the $y-$direction. The two types of directed edges can be viewed as taking an "up" or "down" step in the walk.

Let us consider the number of directed paths in this graph from $(0, 0)$ to an arbitrary point (m, n). Suppose each of these paths makes a up steps and b down steps, then $a+b = m$ and $a - b = n$. Verify that the number of paths from $(0, 0)$ to (m, n) is,

$\omega(m, n) := \binom{a+b}{a} = \frac{(a+b)!}{a!b!} = \frac{m!}{(\frac{m+n}{2})!(\frac{m-n}{2})!}$.

Also note, that the number of paths from (x_1, y_1) to (x_2, y_2) is the same as the number of paths from $(0, 0)$ to $(x_2 - x_1, y_2 - y_1)$.

The remainder of this warm-up exercise uses a classical result due to Gessel and Viennot on the combinatorics of path counting. Please read section 2, "Lindström's theorem" from the paper entitled *Determinants, Paths and Plane Partitions* by Ira M. Gessel and X. G. Viennot.

That is pages 1 and 2 of the paper found here:

`http://people.brandeis.edu/~gessel/homepage/papers/pp.pdf`

For our fundamental example of a digraph, let us choose the weighting so that the weight of every edge is 1. This way, the weight of *all* the paths from u to v is equal to the number of paths from u to v. This means that $P((0,0),(m,n)) = \omega(m,n)$, where $P(u,v)$ is as defined in Gessel and Viennot's paper.

Consider the case of $k = 2$, that is the weight of 2-paths between 2-vertices.

Let $\mathbf{u} = ((0,0),(0,2))$ and $\mathbf{v} = ((2N,0),(2N,2))$.

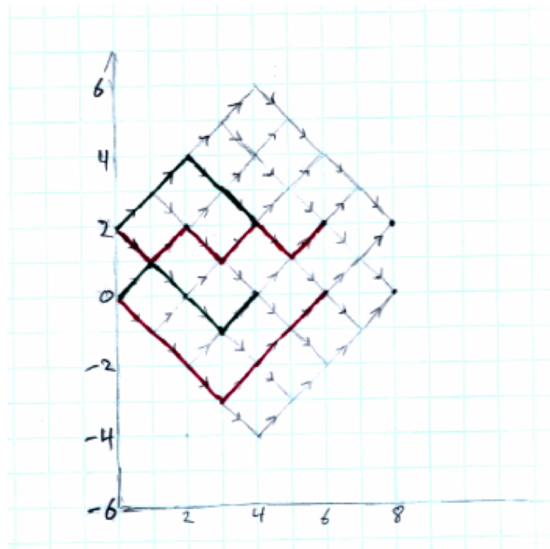
Question: How many "vertex disjoint paths" are there from $\mathbf{u}$ to $\mathbf{v}$? What is $N(\mathbf{u},\mathbf{v})$? (Remember, "vertex disjoint paths" are paths that don't intersect, and don't share any vertices in common either.)

Hint: Use the corollary of Lindström's theorem that you just read about to help you compute this in terms of the previously defined function $\omega(m,n)$*.*

Verify that your formula is correct for $N = 1$ and $N = 2$ by counting the number of vertex-disjoint 2-paths from $\mathbf{u}$ to $\mathbf{v}$ by hand.

Challenge question: Try to think about what changes would need to be made if we were to now consider $k = 3$, and have 3-paths going between $\mathbf{u} = ((0,0),(0,2),(0,4))$ and $\mathbf{v} = ((2N,0),(2N,2),(2N,4))$. Can you write down the matrix, whose determinant would give you the weight of all such 3-paths?

Take a look at the following picture to give you an idea of the situation. The relevant part of the digraph for $k \leq 2$ and $N \leq 4$ is shown. The two green paths are an example of a single vertex-disjoint 2-path where $N = 2$, while the two red paths are an example for $N = 3$.



3 Connecting Aztec diamonds to non-intersecting paths

Why are probabilistic questions about tiling Aztec diamonds related to non-intersecting paths? The reason is that there is a *mapping* which takes each tiling of an $AD(n)$, and produces a *family* of non-intersecting paths. We won't explain the rule here, but rather provide a picture, and ask: can you deduce the rule for producing the non-intersecting paths?