Quantum mechanical properties of Bessel EM modes and their effect on atomic systems

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OBJECTIVES

- Bessel EM modes are studied within the general framework of quantum optics. The basic dynamical operators are identified and their algebraic properties are studied.
- As a mean to measure these dynamical properties, the transition probability for the emission of a Bessel photon by an atomic system is calculated within first order perturbation theory. This permits to analyze the feasibility of observing new rotational effects of twisted light on atoms.

Bessel modes are propagation invariant waves of cylindrical symmetry

Electromagnetic potential in Coulomb gauge

$$\mathbf{A}^{(TM)}(\mathbf{r},t;K) = \frac{c}{i\omega}\mathcal{E}_m^{(TM)}(k_\perp,k_z)\mathbf{N}(\mathbf{r},t;K)$$

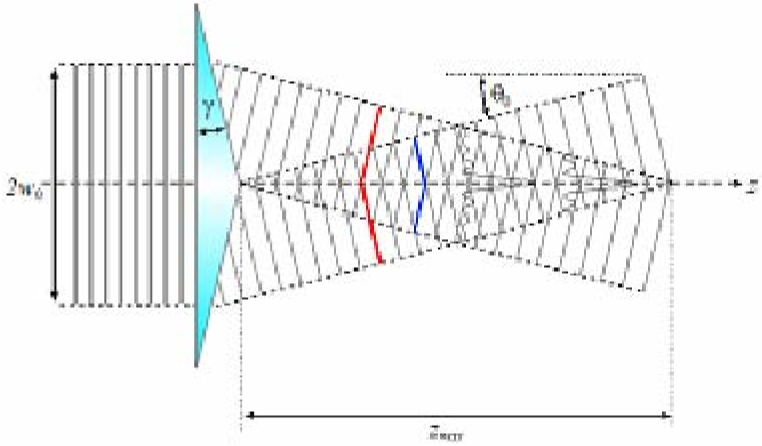
$$\mathbf{A}^{(TE)}(\mathbf{r},t;K) = -\frac{c}{i\omega}\mathcal{E}_m^{(TE)}(k_\perp,k_z)\mathbf{M}(\mathbf{r},t;K),$$

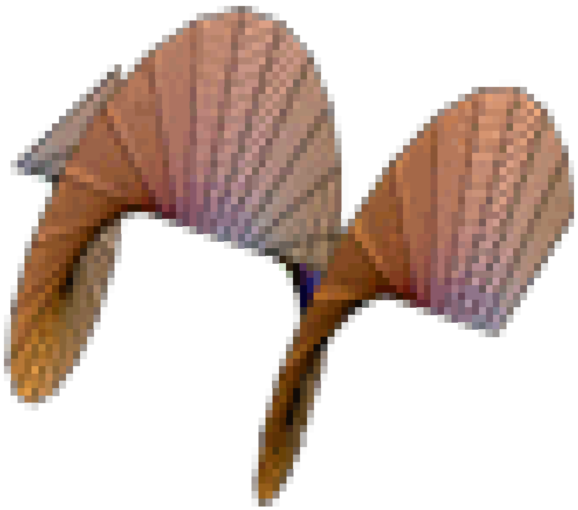
with

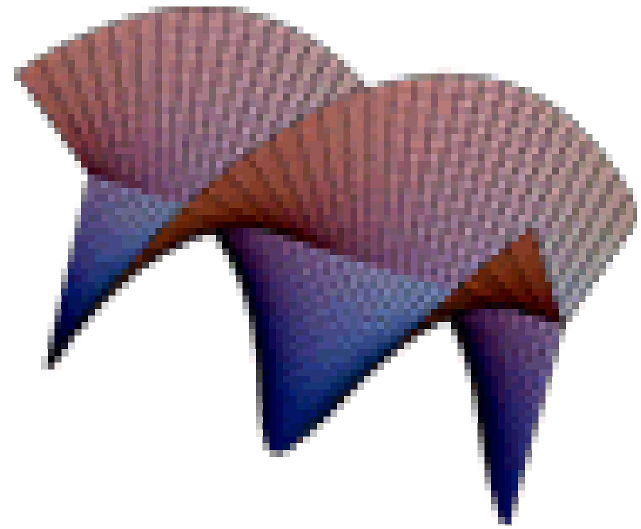
$$\mathbf{M}(\mathbf{r},t;K) \;=\; \frac{\omega}{ck_z}\Big[\frac{m}{k_\perp\rho}J_m(k_\perp\rho)\mathbf{e}_\rho + iJ'_m(k_\perp\rho)\mathbf{e}_\phi\Big] e^{-i\omega t+im\phi+ik_z z}$$

$$\mathbf{N}(\mathbf{r},t;K) \;=\; \Big[iJ'_m(k_\perp\rho)\mathbf{e}_\rho - \frac{m}{k_\perp\rho}J_m(k_\perp\rho)\mathbf{e}_\phi + \frac{k_\perp}{k_z}J_m(k_\perp\rho)\mathbf{e}_z\Big]e^{-i\omega t+im\phi+ik_z z}.$$

•They can be generated approximately using holograms, optical light modulators and other optical dispositives like axicons.

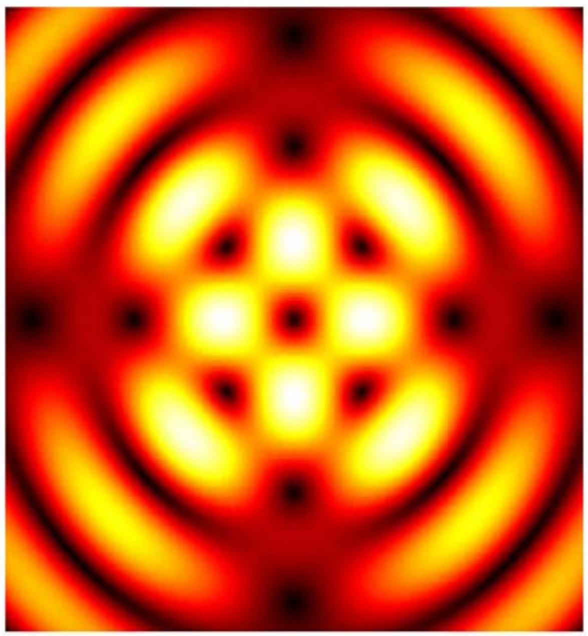


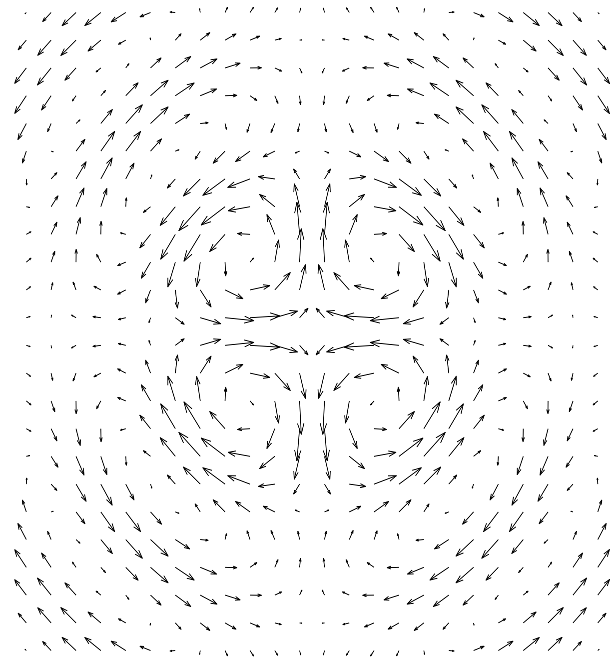


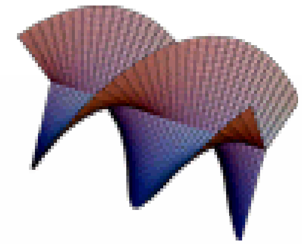


m=1

m=3







Quantization and dynamical variables.

Electromagnetic field operator

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{i=1,2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk_\perp \int_{-\infty}^{\infty} dk_z \big[\hat{a}_m^{(i)}(k_z,k_\perp)\mathbf{A}^{(i)}(\mathbf{r},t;K) + \hat{a}_m^{(i)\dagger}(k_z,k_\perp)\mathbf{A}^{(i)*}(\mathbf{r},t;K)\big], \quad (1)$$

where the annihilation and creation operators satisfy the usual commutation relations

$$[\hat{a}_m^{(i)}(k_\perp,k_z),\hat{a}_{m'}^{(i')\dagger}(k'_\perp,k'_z)] = \delta^{(i,i')}\delta_{m,m'}\delta(k_\perp - k'_\perp)\delta(k_z - k'_z), \quad (2)$$

the index i referring to the two modes of the electromagnetic field, that is, the $TM(i=1)$ and $TE(i=2)$ modes.

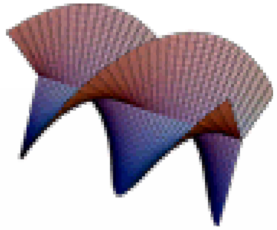
Normalization:

$$\frac{1}{8\pi}\int \big[\mathbf{E}^{(i)*}(\mathbf{r},t;K)\cdot\mathbf{E}^{(i)}(\mathbf{r},t;K') + \mathbf{B}^{(i)*}(\mathbf{r},t;K)\cdot\mathbf{B}^{(i)}(\mathbf{r},t;K')\big]dV = \hbar\omega\delta_{m,m'}\delta(k_\perp - k'_\perp)\delta(k_z - k'_z) \quad , \quad (3)$$

which is equivalent to choosing amplitudes $\mathcal{E}_m^{(TE)}(k_\perp,k_z) = \mathcal{E}_m^{(TM)}(k_\perp,k_z) = k_z c\sqrt{\hbar k_\perp/2\pi\omega}$ for each mode.

Number operator:

$$\hat{N}_m^{(i)} = \frac{1}{2}\Big(\hat{a}_m^{(i)\dagger}\hat{a}_m^{(i)} + a_m^{(i)}\hat{a}_m^{(i)\dagger}\Big), \quad (4)$$



Quantization and dynamical variables.

Quantum energy operator takes the form:

$$\hat{\mathcal{E}} = \hbar \sum_{i,m} \int dk_\perp dk_z \; \omega \; \hat{N}_m^{(i)}(k_\perp, k_z), \tag{1}$$

Momentum operator :

$$\hat{\mathbf{P}}(t) = \frac{1}{8\pi c} \int \left[\hat{\mathbf{E}}(\mathbf{r},t) \times \hat{\mathbf{B}}(\mathbf{r},t) - \hat{\mathbf{B}}(r,t) \times \hat{\mathbf{E}}(\mathbf{r},t) \right] dV. \tag{2}$$

For the Bessel modes under consideration, it takes the form:

$$\hat{\mathbf{P}} = \hbar \sum_{i,m} \int dk_\perp dk_z \left[ik_\perp \hat{a}_{m-1}^{(i)\dagger} \hat{a}_m^{(i)} (\mathbf{e}_x - i\mathbf{e}_y) - ik_\perp \hat{a}_m^{(i)\dagger} \hat{a}_{m-1}^{(i)} (\mathbf{e}_x + i\mathbf{e}_y) + k_z \hat{N}_m^{(i)} \mathbf{e}_z \right]$$

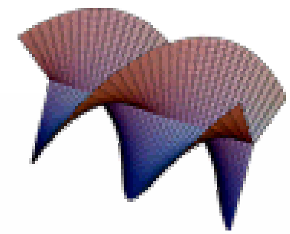
$$= \hbar \sum_i \int dk_\perp dk_z \left[k_\perp \hat{\Pi}_+^{(i)} (\mathbf{e}_x - i\mathbf{e}_y) + k_\perp \hat{\Pi}_-^{(i)} (\mathbf{e}_x + i\mathbf{e}_y) + k_z \hat{\Pi}_3^{(i)} \mathbf{e}_z \right], \tag{3}$$

where the operators $\hat{\Pi}_{\pm,3}^{(i)}(k_\perp, k_z)$ are defined as

$$\hat{\Pi}_+^{(i)} = i \sum_m \hat{a}_{m-1}^{(i)\dagger} \hat{a}_m^{(i)}, \tag{4}$$

$$\hat{\Pi}_-^{(i)} = -i \sum_m \hat{a}_{m-1}^{(i)} \hat{a}_m^{(i)\dagger}, \tag{5}$$

$$\hat{\Pi}_3^{(i)} = \sum_m \hat{N}_m^{(i)}. \tag{6}$$



Quantization and dynamical variables.

Field angular momentum:

$$\mathbf{J}(\mathbf{r_0}) = \frac{1}{4\pi c}\int_{\mathcal{V}}(\mathbf{r}-\mathbf{r}_0)\times[\mathbf{E}(r,t)\times\mathbf{B}(\mathbf{r},t)]dV \quad (1)$$

$$= \mathbf{J}(\mathbf{0}) - \mathbf{r}_0\times\mathbf{P}. \quad (2)$$

Using Maxwell equations, the total angular momentum can also be written as:

$$\mathbf{J}(\mathbf{r}_0) = \frac{1}{4\pi c}\int_{\mathcal{V}} E_i[(\mathbf{r}-\mathbf{r}_0)\times\nabla]A_i\, dV + \frac{1}{4\pi c}\int_{\mathcal{V}} \mathbf{E}\times\mathbf{A}\, dV - \frac{1}{4\pi c}\oint_{\mathcal{S}} \mathbf{E}\big[(\mathbf{r}-\mathbf{r}_0)\times\mathbf{A}\big]\cdot d\mathbf{s}, \quad (3)$$

It is customary to identify with the orbital angular momentum:

$$\mathbf{L}(\mathbf{r}_0) = \frac{1}{4\pi c}\int_{\mathcal{V}} E_i[(\mathbf{r}-\mathbf{r}_0)\times\nabla]A_i\, dV \quad (4)$$

On the other hand,

$$\mathbf{S} = \frac{1}{4\pi c}\int_{\mathcal{V}} \mathbf{E}\times\mathbf{A}\, dV \quad (5)$$

is identified with the spin of the field.

- Gauge dependence. In transverse gauge

$$\nabla \cdot \mathbf{A} = 0,$$

  the results are consistent with the expected values, $\pm\hbar$, of the spin in plane and spherical symmetries.

- In general, the intrinsic angular momentum of a massless particle cannot be defined in an unambiguous way. Instead, the relevant dynamical variable is the helicity and it is actually this quantity that Beth measured in his classical experiment.

- The integral associated to $\mathbf{L}$ is well defined only if the electromagnetic field vanishes fast enough.

For Bessel modes:

$$\hat{\mathbf{L}}(\mathbf{0}) \; = \; \hbar \sum_{i,m} \int dk_\perp dk_z \Big[i\frac{k_z}{k_\perp}(m - \frac{1}{2})\hat{a}^{(i)\dagger}_{m-1}\hat{a}^{(i)}_m \mathbf{e}_- - \; i\frac{k_z}{k_\perp}(m - \frac{1}{2})\hat{a}^{(i)\dagger}_m \hat{a}^{(i)}_{m-1}\mathbf{e}_+ + m\hat{N}^{(i)}_m \mathbf{e}_3 \Big]$$

$$= \hbar \sum_i \int dk_\perp dk_z \Big[\frac{k_z}{k_\perp}\hat{\Lambda}^{(i)}_+ \mathbf{e}_- + \frac{k_z}{k_\perp}\hat{\Lambda}^{(i)}_+ \mathbf{e}_+ + \hat{\Lambda}^{(i)}_3 \mathbf{e}_3, \Big],$$

where the operators $\hat{\Lambda}_{\pm,3}(k_\perp, k_z)$ are given by

$$\hat{\Lambda}^{(i)}_+ \; = \; i\sum_m (m - \frac{1}{2})\hat{a}^{(i)\dagger}_{m-1}(k_\perp, k_z)\hat{a}^{(i)}_m(k_\perp, k_z),$$

$$\hat{\Lambda}^{(i)}_+ \; = \; -i\sum_m (m - \frac{1}{2})\hat{a}^{(i)\dagger}_m(k_\perp, k_z)\hat{a}^{(i)}_{m-1}(k_\perp, k_z),$$

$$\hat{\Lambda}^{(i)}_3 \; = \; \sum_m m\hat{N}^{(i)}_m(k_\perp, k_z).$$

- $\hat{L}_z(\mathbf{0})$ is invariant under Lorentz transformations along the z axis as expected;
- it can be interpreted as an intrinsic operator since it does not depend explicitly on $k_\perp$.
- $\hat{L}_{x,y}(\mathbf{0})$ are highly dependent on quantum numbers $\{k_\perp, m, k_z\}$; moreover, if we define $\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y$, then $\hat{L}_+$ ($\hat{L}_-$) acts as a *lowering* (*rising*) operator that changes $m \rightarrow m - 1$ ($m \rightarrow m + 1$).

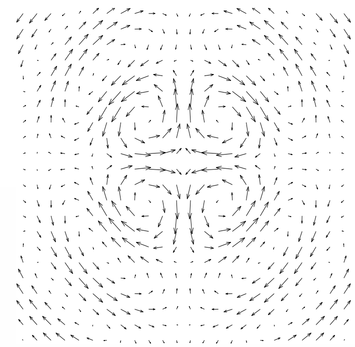
For the helicity operator $\hat{\mathbf{S}}$:

$$\begin{aligned}\hat{\mathbf{S}} &= \hbar\sum_m \int dk_\perp dk_z \frac{c}{2\omega}\big[k_\perp\big(\hat{a}^{(1)}_{m-1}\hat{a}^{(2)\dagger}_{m} - \hat{a}^{(1)\dagger}_{m}\hat{a}^{(2)}_{m-1}\big)\mathbf{e}_- \\ &+ k_\perp\big(\hat{a}^{(1)\dagger}_{m-1}\hat{a}^{(2)}_{m} - \hat{a}^{(1)}_{m}\hat{a}^{(2)\dagger}_{m-1}\big)\mathbf{e}_+ + ik_z\big(\hat{a}^{(1)\dagger}_{m}\hat{a}^{(2)}_{m} \\ &- \hat{a}^{(1)}_{m}\hat{a}^{(2)\dagger}_{m}\big)\mathbf{e}_3\big]\end{aligned}$$

$$= \hbar\int dk_\perp dk_z \frac{c}{\omega}\big[k_\perp\hat{\Sigma}_+\mathbf{e}_- + k_\perp\hat{\Sigma}_-\mathbf{e}_+ + k_z\hat{\Sigma}_3\mathbf{e}_3\big],$$

where the operators $\hat{\Sigma}_{\pm,3}(k_\perp, k_z)$ are defined as

$$\begin{aligned}\hat{\Sigma}_+ &= \frac{1}{2}\sum_m(\hat{a}^{(2)\dagger}_{m}\hat{a}^{(1)}_{m-1} - \hat{a}^{(1)\dagger}_{m}\hat{a}^{(2)}_{m-1}),\\ \hat{\Sigma}_- &= \frac{1}{2}\sum_m(\hat{a}^{(1)\dagger}_{m-1}\hat{a}^{(2)}_{m} - \hat{a}^{(2)\dagger}_{m-1}\hat{a}^{(1)}_{m}),\\ \hat{\Sigma}_3 &= i\sum_m(\hat{a}^{(1)\dagger}_{m}\hat{a}^{(2)}_{m} - \hat{a}^{(1)}_{m}\hat{a}^{(2)\dagger}_{m}).\end{aligned}$$



Algebraic properties of the dynamical operators.

$$[\hat{P}_i, \hat{P}_j] = 0$$

The components of $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ *do not* satisfy the commutation relations of angular momentum. In fact, it can be seen that:

$$\left[\hat{L}_+, \hat{L}_3\right] = \hbar \hat{L}_+, \quad (1)$$

$$\left[\hat{L}_-, L_3\right] = -\hbar \hat{L}_-, \quad (2)$$

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar^2 \sum_i \int dk_\perp dk_z \frac{k_z^2}{k_\perp^2} \hat{\Lambda}_3. \quad (3)$$

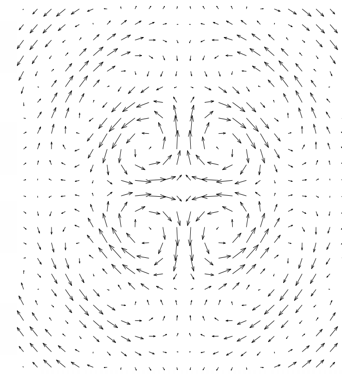
$$[\hat{S}_i, \hat{S}_j] = 0, \quad (4)$$

$$[\hat{P}_i \ , \hat{S}_j] = 0. \quad (5)$$

These properties are compatible with the identification of $\hat{\mathbf{S}}$ as an helicity operator.

$\hat{L}_3$ commutes with the z-component of the linear momentum operator $\hat{\mathbf{P}}$,

$$[\hat{L}_3, \hat{P}_3] = 0. \quad (6)$$



Algebraic properties of the dynamical operators.

while

$$[\hat{L}_3\ ,\ \hat{P}_-] = \hbar \hat{P}_-,$$
$$[\hat{L}_+\ ,\ \hat{P}_-] = \hbar \hat{P}_3,$$
$$[\hat{L}_+\ ,\ \hat{P}_3] = 0,$$
$$[\hat{L}_+\ ,\ \hat{P}_+] = \hbar^2 \sum_{i,m} \int dk_\perp dk_z k_z \hat{a}^{(i)\dagger}_{m-1} \hat{a}_{m+1},$$

and therefore the algebra of these operators does not close.

Finally, $\hat{S}_3$ commutes with $\hat{\mathbf{L}}$,

$$[\hat{S}_3, \hat{\mathbf{L}}] = 0, \tag{1}$$

while

$$\left[\hat{S}_+, \hat{L}_+\right] = -\hbar \hat{S}_z \tag{2}$$
$$\left[\hat{S}_+, \hat{L}_z\right] = -\hbar^2 \int dk_\perp dk_z \frac{ck_\perp}{\omega} \Sigma_+ \tag{3}$$
$$\left[\hat{S}_+, \hat{L}_-\right] = -i\hbar^2 \int dk_\perp dk_z \frac{ck_z}{\omega} \left(a^{(1)}_{m-1} a^{(2)\dagger}_{m+1} - a^{(2)}_{m-1} a^{(1)\dagger}_{m+1} \right).$$

- The components of the momentum operator $\mathbf{P}$ commute among themselves, as it should be, but the algebra they generate with $\hat{\mathbf{L}}(\mathbf{0})$ and $\hat{\mathbf{S}}$ is *not* the standard one for the translation and rotation group.

- Stokes operators:

$$\begin{aligned}
\hat{\sigma}_1 &= \hat{a}^{(TE)\dagger}\hat{a}^{(TM} + \hat{a}^{(TM)\dagger}\hat{a}^{(TE)}, \\
\hat{\sigma}_2 &= i(\hat{a}^{(TM)\dagger}\hat{a}^{(TE)} - \hat{a}^{(TE)\dagger}\hat{a}^{(TM)}), \\
\hat{\sigma}_3 &= \hat{a}^{(TE)\dagger}\hat{a}^{(TE)} - \hat{a}^{(TM)\dagger}\hat{a}^{(TM)}, \\
\hat{\sigma}_0 &= \hat{a}^{(TE)\dagger}\hat{a}^{(TE)} + \hat{a}^{(TM)\dagger}\hat{a}^{(TM)},
\end{aligned}$$

satisfy the algebra of the rotation group: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ up to a factor 2.

- $\hat{L}_3$ and $\hat{S}_3$ as observables. Since $\hat{\mathcal{E}}$, $\hat{P}_3$, $\hat{L}_3(\mathbf{0})$, and $\hat{S}_3$ commute among themselves, they can be simultaneously diagonalized.

$$\hat{a}_m^{(\pm)} =: \frac{1}{\sqrt{2}}\Big(\hat{a}_m^{(1)} \pm i\hat{a}_m^{(2)}\Big),$$

which corresponds to a new basis

$$\mathbf{A}_m^{(\pm)} = \frac{1}{\sqrt{2}}\Big(\mathbf{A}_m^{(TM)} \pm i\mathbf{A}_m^{(TE)}\Big).$$

- Comparison with usual definition of polarized Bessel beams:

$$\mathbf{A}_m^{(\mathcal{R})}(\mathbf{r},t;k_\perp,k_z) = A_0^{(\mathcal{R})}\Big[\mathbf{e}_-\psi_m + \frac{i}{2}\Big(\frac{k_\perp}{k_z}\Big)\psi_{m-1}\mathbf{e_3}\Big],$$
$$\mathbf{A}_m^{(\mathcal{L})}(\mathbf{r},t;k_\perp,k_z) = A_0^{(\mathcal{L})}\Big[\mathbf{e}_+\psi_m - \frac{i}{2}\Big(\frac{k_\perp}{k_z}\Big)\psi_{m+1}\mathbf{e_3}\Big],$$

where $\psi_m(\mathbf{r},t;k_\perp,k_z) = J_m(k_\perp\rho)\exp\{-i\omega t + ik_z z + im\phi\}$. In terms of elementary TE and TM modes:

$$\mathbf{A}_m^{(\mathcal{R})} = A_0^{(\mathcal{R})\prime}\Big(\mathbf{A}_{m-1}^{(TM)} + i\frac{ck_z}{\omega}\mathbf{A}_{m-1}^{(TE)}\Big),$$
$$\mathbf{A}_m^{(\mathcal{L})} = A_0^{(\mathcal{L})\prime}\Big(\mathbf{A}_{m+1}^{(TM)} - i\frac{ck_z}{\omega}\mathbf{A}_{m+1}^{(TE)}\Big).$$

Within the quantization scheme, this change of basis corresponds to the following definition of the annihilation operators:

$$\hat{a}_{m+1}^{(\mathcal{R})} =: \frac{1}{\sqrt{1+(ck_z/\omega)^2}}\Big(\hat{a}_m^{(1)} + i\frac{ck_z}{\omega}\hat{a}_m^{(2)}\Big),$$
$$\hat{a}_{m-1}^{(\mathcal{L})} =: \frac{1}{\sqrt{1+(ck_z/\omega)^2}}\Big(\hat{a}_m^{(1)} - i\frac{ck_z}{\omega}\hat{a}_m^{(2)}\Big).$$

Now, the point is that, although the helicity operator $\hat{S}_3$ is diagonal in this basis:

$$\hat{S}_3 = \hbar\sum_m \int dk_\perp dk_z \frac{1+(\omega/ck_z)^2}{2}\Big(\hat{\mathcal{N}}_m^{(\mathcal{R})} - \hat{\mathcal{N}}_m^{(\mathcal{L})}\Big),$$

the operators $\hat{\mathcal{E}}$, $\hat{P}_3$ and $\hat{L}_3$ are not diagonal. It should be noticed that it is only in the paraxial approximation, $k_z \sim \omega/c$, that the second term in this last equation, which is non diagonal, does vanish.

$\hat{L}_3$ and $\hat{S}_3$ as observables. Since $\hat{\mathcal{E}}$, $\hat{P}_3$, $\hat{L}_3(\mathbf{0})$, and $\hat{S}_3$ commute among themselves, they can be simultaneously diagonalized.

Interaction of Bessel photons with atomic systems as a means to measure these observables.

$$\hat{H} = \hat{H}_P + \hat{H}_R + \hat{H}_I,$$

with $\hat{H}_I$

$$\begin{aligned}
\hat{H}_I &= \hat{H}_{I1} + \hat{H}_{I2} + \hat{H}_{I3} \\
\hat{H}_{I1} &= -\sum_{i=1}^{2} \frac{q_i}{M_i} \mathbf{p}_i \cdot \hat{\mathbf{A}}(\mathbf{r}_i) \\
\hat{H}_{I2} &= \sum_{i=1}^{2} \frac{q_i^2}{2M_i} |\hat{\mathbf{A}}(\mathbf{r}_i)|^2 \\
\hat{H}_{I3} &= \sum_{i=1}^{2} g_i \frac{q_i}{2M_i} \mathbf{S}_i \cdot \hat{\mathbf{B}}(\mathbf{r}_i).
\end{aligned}$$

$\mathbf{S}_i$ denotes the spin particle i.

Interaction Hamiltonian H_{I1}.

For $m > 0$

$$\begin{aligned}
\xi_{mm}(\rho,\varphi;k_\perp) &= J_l(k_\perp\rho)e^{im\varphi} \\
&= 2^m(m-1)!\sum_{v=0}^{\infty}(m-v)\frac{J_{m+v}(k_\perp R_\perp)J_{m+v}(k_\perp q_\perp)}{(k_\perp q_\perp)^m} \\
&\cdot \sum_{s=0}^{v}\frac{\Gamma(m+s)\Gamma(m+v-s)}{s!(v-s)!(\Gamma(m))^2}\cos((v-2s)(\varphi_R-\varphi_q)) \\
&\cdot \sum_{n=0}^{m}(-1)^n\binom{m}{n}\left(\frac{q_\perp}{R_\perp}\right)^n e^{i(m-n)\varphi_R}e^{in\varphi_q},
\end{aligned}$$

For $m = 0$

$$\psi_0(\rho,\varphi;k_\perp) = J_0(k_\perp\rho) = \sum_{v=-\infty}^{\infty} J_v(k_\perp R_\perp)J_v(k_\perp q_\perp)\cos(v(\varphi_R-\varphi_q)).$$

$$\vec{\rho} = \mathbf{R}_\perp - \mathbf{q}_\perp.$$

Unperturbed wave function

$$\Phi(\mathbf{R}) = \frac{1}{\sqrt{2\pi}} e^{i m_R \varphi_R} \Upsilon_{CM}(R_\perp, z_R)$$
$$\phi(\mathbf{r}) = \Theta(r) Y_{l_r m_r}(\theta, \varphi_r)$$

with Y_{lm} the spherical harmonics.

Selection rules

$$\delta(m - n \pm v \mp 2s - m_R + m'_R)\delta(n \mp v \pm 2s - m_r + m'_r)$$

for the transition amplitudes proportional to $E^{(0)}_{CM} - E^{(F)}_{CM}$, and

$$\delta(m - i - n \pm v \mp 2s - m_R + m'_R)\delta(i + n \mp v \pm 2s - m_r + m'_r)$$

with $i = \pm 1, 0$, for the transition amplitudes proportional to $E^{(0)}_{rel} - E^{(F)}_{rel}$.

Here the letters n, s and v denote the summation indices as they appear in multipolar equation.

The total change in the projection of the angular momentum of the atom along the z axis **is always** $-m\hbar$.

For $k_\perp \rho_r \ll 1$ the term $v = 0$ is expected to be dominant in the series expansion of the vector potential and the functions ψ_m can be approximated by

$$\psi_m \sim e^{ik_z z_R - i\omega t} J_m(k_\perp R_\perp) \sum_{n=0}^{m} (-1)^n \binom{m}{n} \left(\frac{q_\perp}{R_\perp}\right)^n e^{i(m-n)\varphi_R} e^{in\varphi_q}.$$

Besides, if the atom is located outside the axes of the Bessel beam, in general, $q_\perp \ll R_\perp$ and the $n = 0$ term is dominant. Under such conditions the neutral atom ($q_e = -q_N$) transition amplitude

$$\begin{aligned}\langle F, 1_K^{(i)} | H_{I1} | 0; 0 \rangle \;\sim\; & \frac{q_e}{i\hbar}(E^0_{rel} - E^F_{rel}) \int d^3R \Phi_0^*(\mathbf{R}) \mathbf{A}_K^{(i)*}(\mathbf{R}) \Phi_F(\mathbf{R}) \\ & \cdot \int d^3r \phi_F^*(\mathbf{r}) \mathbf{r} \phi_0(\mathbf{r}).\end{aligned}$$

contains the standard dipole matrix element for the relative coordinates.

$$\langle F, 1_K^{TE}|H_{I1}|0;0\rangle \sim \frac{q_e E_0}{k_z \hbar}(E^0_{rel} - E^F_{rel})e^{-i(\omega t - (E_F - E_0)t/\hbar)}$$

$$\cdot \sum_{j=\pm 1} \delta_{m-j, m_R - m'_R} \delta_{j, m'_r - m_r} I^{(0)}_{CM}(k_\perp, k_z, m-j) I_{rel}(k_\perp, k_z, j)$$

$$\langle F, 1_K^{TM}|H_{I1}|0;0\rangle \sim \frac{q_e E_0 c}{i\hbar\omega}(E^0_{rel} - E^F_{rel})e^{-i(\omega t - (E_F - E_0)t/\hbar)}$$

$$\cdot \Big[-\frac{2k_\perp i}{k_z}\big(\delta_{m, m'_R - m_R}\delta_{m'_r, m_r} I^{(0)}_{CM}(k_\perp, k_z, m) I_{rel}(k_\perp, k_z, 0)$$

$$+ \sum_{j=\pm 1} (-j)\delta_{m-j, m_R - m'_R} \delta_{j, m'_r - m_r} I^{(0)}_{CM}(k_\perp, k_z, m-j) I_{rel}(k_\perp, k_z, j)\big)\Big]$$

with

$$I^{(0)}_{CM}(k_\perp, k_z, m-j) = \int dR_\perp dz \Upsilon^{F*}_{CM}(R_\perp, z) e^{ik_z z} J_{m-j}(k_\perp R_\perp) \Upsilon^0_{CM}(R_\perp, z),$$

and

$$I_{rel}(k_\perp, k_z, j) = \frac{1}{2l'_r + 1}\Big[\delta_{j,0}[(l'_r - |m_r| + 1)\delta_{l_r, l'_r + 1} + (l'_r - |m_r| - 1)]$$

$$+(1 - \delta_{j,0})[\delta_{l_r, l'_r+1}\delta_{|m_r|, |m'_r| - 1} - \delta_{l_r, l'_r - 1}\delta_{|m_r|, |m'_r| + 1}]\Big] \int dr r^3 \Theta^*_F(r)\Theta_0(r).$$

- Selection rules for the transitions involving the relative motion of the charged particles are the same as those obtained with a plane wave expansion of the radiation potential.

- Emission of a Bessel photon of orbital angular momentum $m\hbar$ yielding $m_r' = m_r \pm 1$ leads to a rotational recoil effect $m_R' = m_R - m \mp 1$ for the center of mass. While, to this order of approximation, transitions with $m_r' = m_r$ and $m_R' = m_R - m$ are allowed just for the emission of transverse magnetic photons. These transitions favor torque effects on the center of mass and are relevant in the emission of non paraxial photons.

- With the above formula we can now calculate the corresponding transition amplitudes for Bessel photons polarized in such a way that they are also eigenfunctions of the helicity operator $\hat{S}_3$. The result is

$$\langle F, 1_K^{\pm}|H_{I1}|0;0\rangle \sim \frac{q_e i\sqrt{2k_\perp}}{\sqrt{h\omega}}(E_{rel}^0 - E_{rel}^F)e^{-i(\omega t-(E_F-E_0)t/\hbar)}$$

$$\cdot\Big[-\frac{ick_\perp}{\omega}\delta_{m,m'_R-m_R}\delta_{m'_r,m_r}I_{CM}^{(0)}(k_\perp,k_z,m)I_{rel}(k_\perp,k_z,0)\pm$$

$$\sum_{j=\pm1}\left(\frac{1\pm jk_zc/\omega}{2}\right)\delta_{m-j,m_R-m'_R}\delta_{j,m'_r-m_r}I_{CM}^{(0)}(k_\perp,k_z,m-j)I_{rel}(k_\perp,k_z,j)\Big].$$

The value of the helicity factor $\pm k_zc/\omega$ determine directly the transition amplitudes so that transitions that change the projection of the angular momentum of the internal atomic motion by $\pm\hbar$ may be directly enhanced or suppressed by modifying its value.

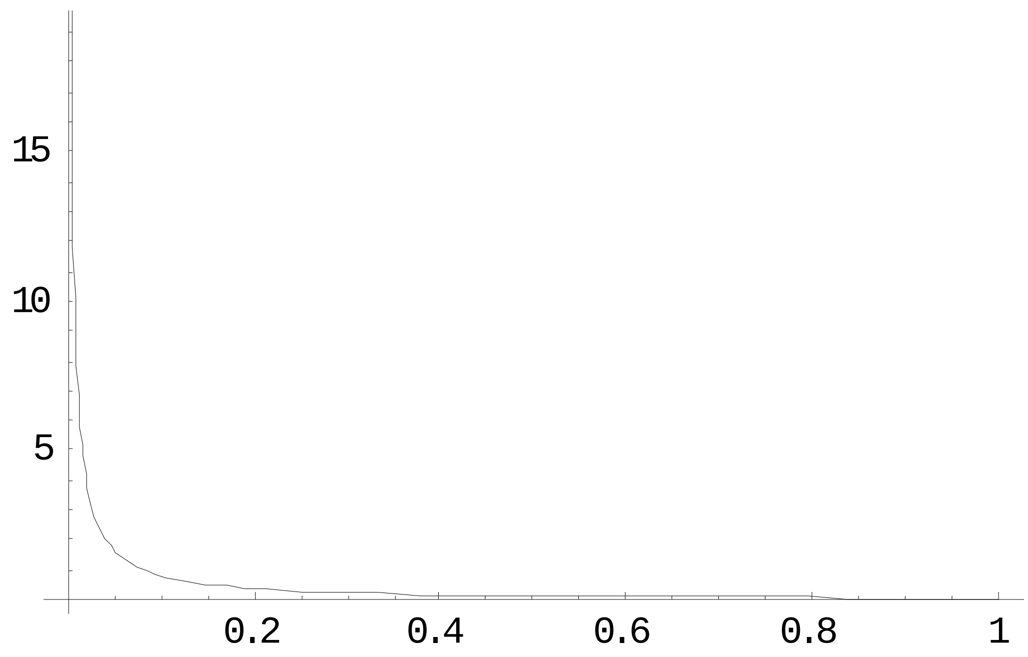
The terms with $n > 0$, are expected to be relevant only when the atomic center of mass is located close to the axis of symmetry of the Bessel mode. However, at that axis there is a vortex of charge m so that $J_m(k_\perp \rho) = 0$ for $\rho = 0$ if $m \neq 0$ and even if the center of mass is properly located, the matrix element is expected to be small.

For trapped atoms, the transverse part of the center of mass wave function is localized. The transition amplitude depends on the average position of the center of mass and on the spread of the oscillation α.

$$\begin{aligned}\mathcal{J}(\bar{n}, \alpha, k_\perp, m) &= (\sqrt{\alpha})^{n-m} \int_0^\infty R_\perp^{m-2n+1} e^{\rho_\perp^2/\alpha^2} J_m(k_\perp R_\perp) L_{\bar{n}}^{m-n}(R_\perp^2/\alpha^2) \\ &= \frac{k_\perp^{m+2n}}{2^{m+2n+1}\bar{n}!(\sqrt{\alpha})^{-\bar{n}-1}} \sum_{r=0}^{\infty} \frac{(m+\bar{n}-n+r)!}{(m+\bar{n}+r)!r!} (\frac{-k_\perp^2 \alpha^2}{4})^r.\end{aligned}$$

Ejemplo:
m=2, n=2

$$\frac{e^{-x}\left(6-6e^{x}+4x+2e^{x}x+x^{2}\right)}{x^{4}}$$



Matrix elements of the interaction term H_{I2}.

- Two photons transitions.
- $q_e^2/2M_e \gg q_N^2/2M_N$

$$\hat{H}_{I2} \sim \frac{q_e^2}{2M_e}|\hat{\mathbf{A}}(\mathbf{R}+\frac{\mu}{M_e}\mathbf{r})|^2. \tag{1}$$

- Long wavelength limit

$$\psi_m(\mathbf{R}_\perp+\frac{\mu}{M_e}\mathbf{r}_\perp,z)\psi_{m'}(\mathbf{R}_\perp+\frac{\mu}{M_e}\mathbf{r}_\perp,z) \sim e^{i(k_z+k'_z)z_R-i(\omega+\omega')t}J_m(k_\perp R_\perp)J_{m'}(k'_\perp R_\perp)$$

$$\cdot \sum_{n,n'=0}^{m}\binom{m}{n}\binom{m'}{n'}\Big(-\frac{\mu}{M_e}\frac{r_\perp}{R_\perp}\Big)^{(n+n')}e^{i(m+m'-n-n')\varphi_R}e^{i(n+n')\varphi_r}. \tag{2}$$

Unless the atom is located on the beam axis, it is expected that the most important contributions to the transition probabilities come from the $n=n'=0$ terms in these series. The two photons are then emitted producing a translational and rotational recoil effect on the center of mass while the internal state of the atom remains invariant.

Matrix elements of the interaction term H_{I3}

- The interaction term $\hat{H}_{I3}$ opens the possibility to change the spin of the particles.

$$g_i \frac{q_i}{2M_i} \mathbf{S}_i \cdot \hat{\mathbf{B}}^{(TM)}(\mathbf{r}_i) = g_i \frac{q_i \omega}{4M_i c k_z} E_0 \left[\psi_{m-1}(\mathbf{r}_i) \hat{S}_+^{(i)} + \psi_{m+1}(\mathbf{r}_i) \hat{S}_-^{(i)} \right] \quad (3)$$

$$g_i \frac{q_i}{2M_i} \mathbf{S}_i \cdot \hat{\mathbf{B}}^{(TE)}(\mathbf{r}_i) = g_i \frac{i q_i}{4M_i} E_0 \left[\psi_{m-1}(\mathbf{r}_i) \hat{S}_+^{(i)} - \psi_{m+1}(\mathbf{r}_i) \hat{S}_-^{(i)} - \frac{2ik_\perp}{k_z} \psi_m(\mathbf{r}_i) \hat{S}_z^{(i)} \right]. \quad (4)$$

with $S_\pm$ the ascending and descending particle spin operators.

- The most probable event leads to changes of the spin angular momentum of the electron in a factor $\pm\hbar$ without changes in the relative motion spatial wave function, while the center of mass acquires an angular momentum $-(m \mp 1)\hbar$. The event corresponding to no changes in total internal wave function at the expense of a recoil effect with a change of the space angular momentum of the center of mass in $-m\hbar$ is also relevant for non paraxial photons.

- The matrix elements of the center of mass calculated for the H_{I1} can also be used in the evaluation of the transition amplitudes associated to H_{I3}.

Conclusions

Quantization of Bessel EM modes.

- The proper values of the set of observables $\{\hat{\mathcal{E}}, \hat{P}_3, \hat{L}_3(\mathbf{0}), \hat{S}_3\}$ define the possible quantum numbers that characterize the Bessel photons: $\{\omega, \hbar k_z, m\hbar, \pm\hbar k_z c/\omega\}$.

- The three components of the operator $\mathbf{L} = \{\hat{L}_+, \hat{L}_-, \hat{L}_3\}$ do not satisfy a closed algebra, despite the fact that $\hat{L}_3$ is related to a spatial rotation around the z axis.

- This algebra is not the same as that obtained for localized fields by S. J. van Enk and G. Nienhuis.

- The algebra of the local operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ is independent of the gauge and the basis set. Accordingly, global bilinear operators of the electromagnetic fields such as $\hat{\mathbf{P}}$ and $\hat{\mathbf{S}}$ have commutation relations among all their components that are also independent of the basis set; this is guaranteed by the normalization condition that each photon carries an energy $\hbar\omega$. However, the global operators $\hat{\mathbf{J}}$ and $\hat{\mathbf{L}}$ are defined in terms not only of $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$, but also of the position vector $\mathbf{r}$; this term induces a strong dependence of $\hat{\mathbf{J}}$ and $\hat{\mathbf{L}}$ on the boundary conditions satisfied by $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$.

- The algebraic properties of the dynamical operators and their commutation relations have physical consequences because they imply, for instance, specific uncertainty relations that could be verified experimentally.

- All the dynamical operators we have studied correspond to global observable quantities. A further analysis of local dynamical quantities, such as the tensor M_{ij} describing the angular momentum flux, could elucidate the difference between "spin" and "orbital" angular momentum. In fact, it was shown by Barnett that in the classical case, there is a natural separation into spin and orbital parts for the z component of this flux, M_{zz}. However, a quantum description should also include the full commutation relations of the appropriate separated parts of this tensor. This is particularly relevant in the light of recent experiments measuring the rates of spin and orbital rotation of trapped particles at different distances from the beam axis.

- The fact that spontaneously emitted optical Bessel photons have not been observed can be due to the small value of the center of mass matrix elements under usual circumstances.

- For trapped atoms, the transition amplitude depends on the average position of the center of mass and on the spread of the oscillation α. We have shown that the matrix element of the center of mass motion that goes together with the standard electric dipole matrix for the internal motion, , decays exponentially with the factor $k_\perp^2\alpha^2/4$ relating the spread of the atom oscillation to the photon wavelength.

- We have focused our attention on spontaneous transition amplitudes but induced transition probabilities can be calculated from them. They will be proportional to incident radiation intensity and could experimentally confirm our results. We have explicitly shown that certain mechanisms can enhance the probability of changing the internal angular momentum of the atom in multiples of $\hbar$ bigger than those predicted by the standard plane wave multipole expansion. For instance, transitions with $\Delta m^r = \pm 2\hbar$ depend on the electric quadrupole matrix elements of the relative motion and are given by two kind of transition amplitudes: one for the standard quadrupole expansion that is proportional to $k_\perp$ and the other arising from terms with $n = 1$ and $k = 0$. The latter are due to the vortex of the Bessel mode in the beam axes and could be observable for a trapped atom for an adequate value of $k_\perp^2 \alpha^2/4$.

- Transition amplitudes associated to the interaction Hamiltonian

$$\hat{H}_{I2} = \sum_{i=1}^{2} \frac{q_i^2}{2M_i} |\hat{\mathbf{A}}(\mathbf{r}_i)|^2.$$

  In the long wavelength limit, the most important transitions associated to the former Hamiltonian involve two photons that do not alter the internal motion of the atom but exchange linear $\hbar(k_z + k_z')$ and angular momentum $(m + m')\hbar$ with the center of mass.

- Transition amplitudes associated to the interaction Hamiltonian

$$\hat{H}_{I3} = \sum_{i=1}^{2} g_i \frac{q_i}{2M_i} \mathbf{S}_i \cdot \hat{\mathbf{B}}(\mathbf{r}_i).$$

  For hydrogenic atoms, the magnetic interaction $g_i q_i / 2M_i \mathbf{S}_i \cdot \hat{\mathbf{B}}(\mathbf{r}_i)$ will favor changes in the spin orientation of the electron $\pm\hbar$ and in the orbital angular momentum of the center of mass.

- The quantum electrodynamic interaction of atoms with Bessel beams clarify the role of $\hat{S}_3$ and $\hat{L}_3$ as observables. The atom can be regarded as a measurement instrument. Absorption or emission rates of elementary Bessel modes $\mathbf{A}(K) = \mathbf{A}^{(TM)}(K) \pm i\mathbf{A}^{(TE)}(K)$ are conditioned to a change of $m\hbar$ in the projection along the z axis of the atomic total angular momentum. Besides the helicity factor $\pm k_z c/\omega$ can be used to directly enhance or suppress such transitions.

- The vectorial character of the electromagnetic field is responsible of possible changes $\pm\hbar$ in the internal angular momentum within the dipole approximation. The transition probability of emission of TM Bessel photons via H_{I1} without changes in the internal angular momentum necessarily leads to the maximum possible exchange of angular momentum between a Bessel photon and the center of mass motion.

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