THE NON-LINEAR SCHRODINGER EQUATION AND SOLITONS

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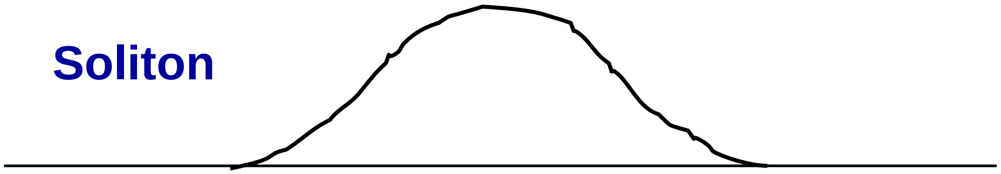
Soliton



John Scott Russell (Solitary Water Wave –1834)

Korteweg and DeVries (KdeV equation – 1895)

Zabusky and Kruskal (Solitons -- 1965)

Zhakharov and Shabat (NLSE – 1971)

Hasegawa and Tappert (Lightwave Solitons – 1973)

Mollenauer et. al. (Observation in fiber – 1980)

THE NONLINEAR SCHRODINGER EQUATION

$$-i\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + |u|^2 u - i(\alpha/2)u$$

chromatic dispersion

based on $n = n_0 + n_2 I$

CLASSICAL SOLITON:

$$u(z,t) = sech\ (t)\ e^{iz/2}$$

SOLITON UNITS:

Length	Time	Power
$z_c = 0.322\frac{2\pi c}{\lambda^2}\frac{\tau^2}{D}$	$\frac{\tau}{1.7627...}$	$P_{sol} = \frac{A_{eff}}{2\pi n_2}\frac{\lambda}{z_c} \propto \frac{D}{\tau^2}$

(For $\tau = 20$ ps, $D = 0.5$ ps/nm-km,
and λ= 1556 nm, z_c ~200 km.)

$$(1.7627... = 2\cosh^{-1}\sqrt{2}\ )$$

PHASE AND GROUP VELOCITIES
Z=VT

Consider a medium characterized by the dispersion relation $\omega(k)$.

The phase of any frequency component is

$$\phi(z,t) = \omega t - kz$$

For an observer in a frame moving at velocity v

$$\frac{d\phi}{dt} = \omega - kv$$

For stationary ϕ, the above = zero.

The corresponding solution, $v_p = \omega/k$, is the phase velocity.

A pulse's peak occurs where its Fourier components add most constructively.

Thus, in a frame moving at the group velocity, v_g, one must have:

$$\frac{d}{dk}\left(\frac{d\phi}{dt}\right) = \frac{d\omega}{dk} - v_g = 0$$

The solution is the well-known result $v_g = d\omega/dk$.

INVERSE GROUP VELOCITY
TIME DELAY = Z/V

Let the dispersion relation be written as $k(\omega)$.

Let the phase of any frequency component be written as

$$\phi(z,t) = kz - \omega t$$

For an observer in a time frame moving at reciprocal velocity v^{-1},

$$\frac{d\phi}{dz} = k - \omega v^{-1}$$

For stationary ϕ, the above = zero.

The corresponding solution, $v_p^{-1} = k/\omega$, is the reciprocal phase velocity.

A pulse's peak occurs where its Fourier components add most constructively.

Thus, in a time frame moving with reciprocal group velocity v_g^{-1}, one must have:

$$\frac{d}{d\omega}\left(\frac{d\phi}{dz}\right) = \frac{dk}{d\omega} - v_g^{-1} = 0$$

The solution is $v_g^{-1} = dk/d\omega$

FIBER NONLINEARITY

The induced polarization in a nonlinear dielectric takes the form:

$$P = \varepsilon_0 \left[\chi^{(1)} \cdot E + \chi^{(2)} : EE + \chi^{(3)} \cdot EEE + \cdots \right]$$

For (isotropic) fibers, $\chi^{(1)} = n^2 - 1$ (n is the index of refraction), while $\chi^{(2)} = 0$.

$\chi^{(3)}$ yields third harmonic generation (ordinarily negligibly weak in silica glass fibers), four-wave mixing, and nonlinear refraction; only the later two are of interest here.

In silica glass fibers, the index can be written, with great accuracy, as:

$$n(\omega, |E|^2) = n(\omega) + n_2 |E|^2$$

where n_2 is related to $\chi^{(3)}$ by

$$n_2 = \frac{3}{8n} \chi^{(3)}_{xxx}$$

($\chi^{(3)}_{xxx}$ is a scalar component of $\chi^{(3)}$, appropriate to the polarization.)

In silica glass fibers, if we write the nonlinear index as $n_2 I$, where I is the intensity, then n_2 has the polarization-averaged value of 2.8×10^{-16} cm^2/W.

DERIVATION OF THE NLS EQUATION

Represent the complex field amplitudes with a scalar, dimensionless fn. $U(z,t)$, where $P = P_c|U|^2$, and let $k(\omega,P)$ be the dispersion relation of monochromatic waves $U(z,t) = u_0 e^{i(kz - \omega t)}$

Expand $k(\omega,P)$ in a Taylor series about $(\omega_0,0)$:

$$k = k_0 + k'(\omega-\omega_0) + ½k''(\omega-\omega_0)^2 + k_2 P \qquad (1)$$

Reciprocal group velocity (time per unit distance):

$$v_g^{-1} = \frac{\partial k}{\partial \omega} = k' + k''(\omega-\omega_0)$$

i.e., $k' = v_g^{-1}(\omega_0)$ and k'' is the dispersion constant.

Note: Dispersion is often quoted as a wavelength derivative:

$$D \equiv \frac{\partial k'}{\partial \lambda} = -\frac{2\pi c}{\lambda^2} k''$$

DERIVATION OF THE NLS EQUATION
Step II: Shift to central frequency and retarded time

Remove ω_0 and k_0 from $U(z,t)$ by defining

$$u(z,t) = U(z,t)\, e^{i(\omega_0 t - k_0 z)} = u_0\, e^{i[(k-k_0)z - (\omega-\omega_0)t]}$$

The equation for u that reproduces the dispersion relation (1) is clearly

$$-i\frac{\partial u}{\partial z} = i\,k'\,\frac{\partial u}{\partial t} - \tfrac{1}{2}\,k''\frac{\partial^2 u}{\partial t^2} + k_2\,P_c\,|u|^2 u \qquad (2)$$

To make the time-frame for representing a pulse ind. of z, transform to retarded time:

$$t_{ret} = t - k'z$$

Since the eqn. of motion for a pulse at ω_0 is $t_{ret} = const.$, we have:

$$k'_{ret} = \frac{dt_{ret}}{dz} = \frac{dt}{dz} - k' = 0,$$

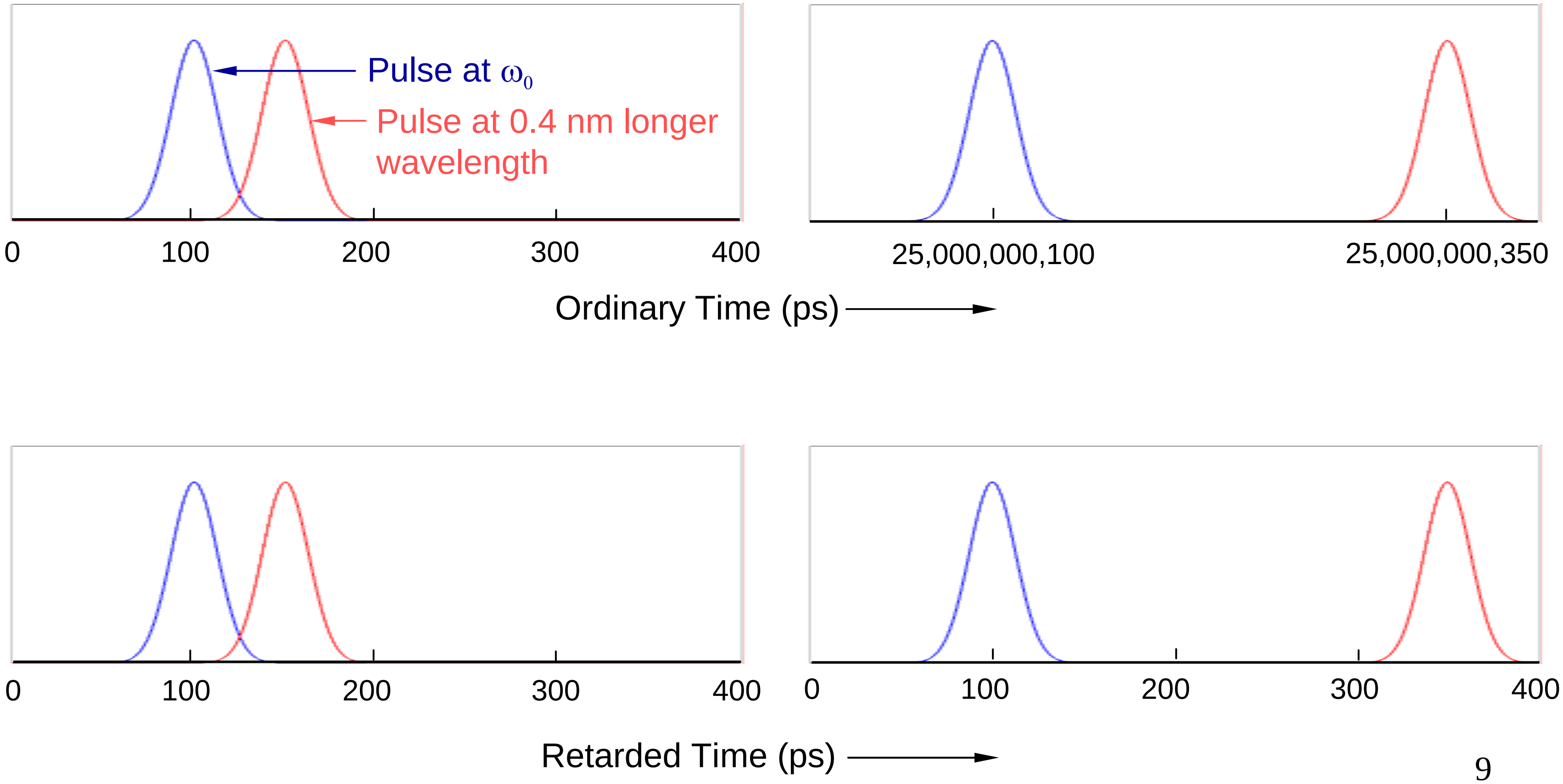
Thus, the (coefficient of the) $\frac{\partial u}{\partial t}$ term in (2) is eliminated.

PULSE REPRESENTATION IN ORDINARY VS RETARDED TIME

Pulses separated by $\delta\lambda$ = 0.4 nm in fiber with D = 0.1 ps/nm-km

Pulses at 0 km:

Pulses at 5000 km:



DERIVATION OF THE NLS EQUATION

Step III: Rescale the independent variables

Choose unit values t_c, z_c, and P_c for time, distance, and power, respectively, such that the rescaled coefficients k'' and $k_2 P_c$ in (2) each become unity. That is, we have the new, "dimensionless" variables

$$t' = t_{ret}/t_c = (t - k'z)/t_c$$

$$z' = z/z_c$$

where the new unit values must obey the relations

$$-t_c^2/z_c = k''$$

$$(z_c P_c)^{-1} = k_2$$

Rewriting eqn. (2) in terms of the new variables, and dropping the primes, we get

$$-i\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + |u|^2 u$$

FOURIER TRANSFORMS

Let $u(t)$ and $\tilde{u}(\omega)$ be Fourier transforms of each other, i.e.,

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{u}(\omega)\, e^{-i\omega t} d\omega \quad \longleftrightarrow \quad \tilde{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(t)\, e^{i\omega t} dt$$

Note that

$$\frac{\partial u(t)}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} - i\omega\, \tilde{u}(\omega)\, e^{-i\omega t} d\omega$$

i.e., $\dfrac{\partial u(t)}{\partial t} \quad \longleftrightarrow \quad - i\omega\, \tilde{u}(\omega)$

similarly, $\dfrac{\partial^2 u(t)}{\partial t^2} \quad \longleftrightarrow \quad - \omega^2 \tilde{u}(\omega)$

THE NLS EQN: ACTION OF THE DISPERSIVE TERM

To get action of dispersive term alone, turn off NL term, so eqn. becomes:

$$\frac{\partial u}{\partial z} = \frac{i}{2}\,\frac{\partial^2 u}{\partial t^2}$$

The problem is most naturally solved in the frequency domain, where eqn. becomes:

$$\frac{\partial \tilde{u}}{\partial z} = -\,\frac{i}{2}\,\omega^2\,\tilde{u}$$

and where the general solution is:

$$\tilde{u}(z,\omega) = \tilde{u}(0,\omega)\; e^{-i\frac{\omega^2 z}{2}}$$

Thus, the dispersive term merely rearranges the phase relations among existing frequency components; it adds no new ones.

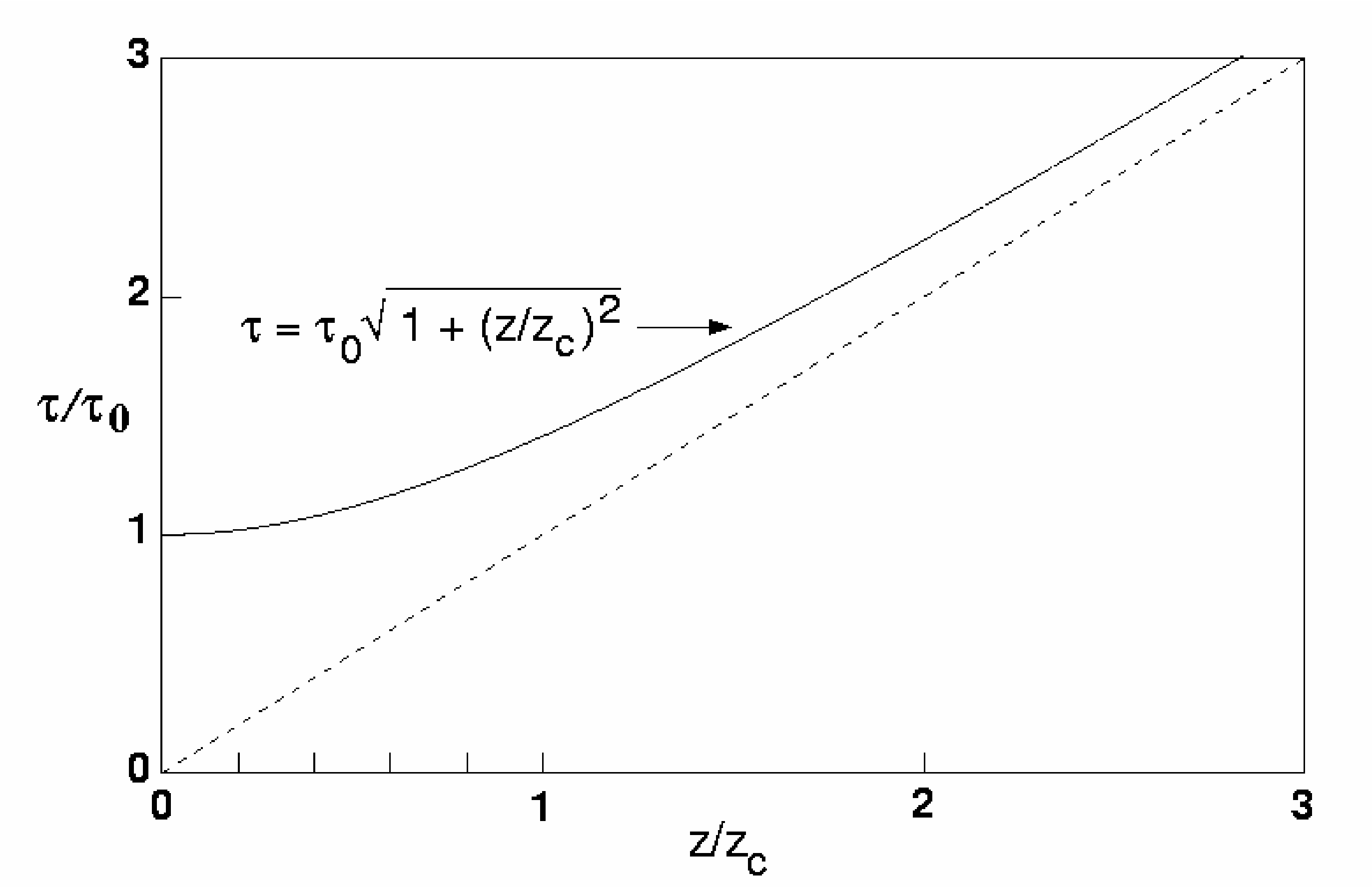
To get the spreading of the pulse, must transform back to the time domain.

Example: Let $u(0,t) = e^{-t^2/2}$ for which $\tilde{u}(0,\omega) = e^{-\omega^2/2}$.

$$u(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{u}(0,\omega)\; e^{-i\frac{\omega^2 z}{2}}\; e^{-i\omega t}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}(1+iz)}\; e^{-i\omega t}\,d\omega = e^{-\frac{t^2}{2(1+iz)}}$$

Thus, intensity envelope $|u|^2 = e^{-t^2/(1+z^2)} = e^{-(2t/\delta t)^2}$, where $\delta t/2 = \sqrt{1+z^2}$.

DISPERSIVE BROADENING OF A GAUSSIAN PULSE VS Z
(Minimum temporal width at origin)



THE NLS EQN: ACTION OF THE NON-LINEAR TERM

To get action of NL term alone, turn off dispersive term, so eqn. becomes:

$$\frac{\partial u}{\partial z} = i|u|^2 u$$

The problem is most naturally solved in the time domain, where gen. soln. is:

$$u(z,t) = u(0,t)\, e^{i|u|^2 z}$$

The NL term modifies $\phi(t)$, but not the intensity envelope.

Thus, it only adds new frequency components.

To get spectral spreading, must transform back to the frequency domain.

Example: Again, let $u(0,t) = e^{-t^2/2}$.

$$\tilde{u}(z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(0,t)\, e^{i|u|^2 z}\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\, e^{iz e^{-t^2}}\, e^{i\omega t}\, d_t$$

For $z >> 1$, this integral produces a multi-peaked spectrum, where the number of peaks and the over-all spectral width increase directly with z.

However, for $z << 1$, the integral is approximately

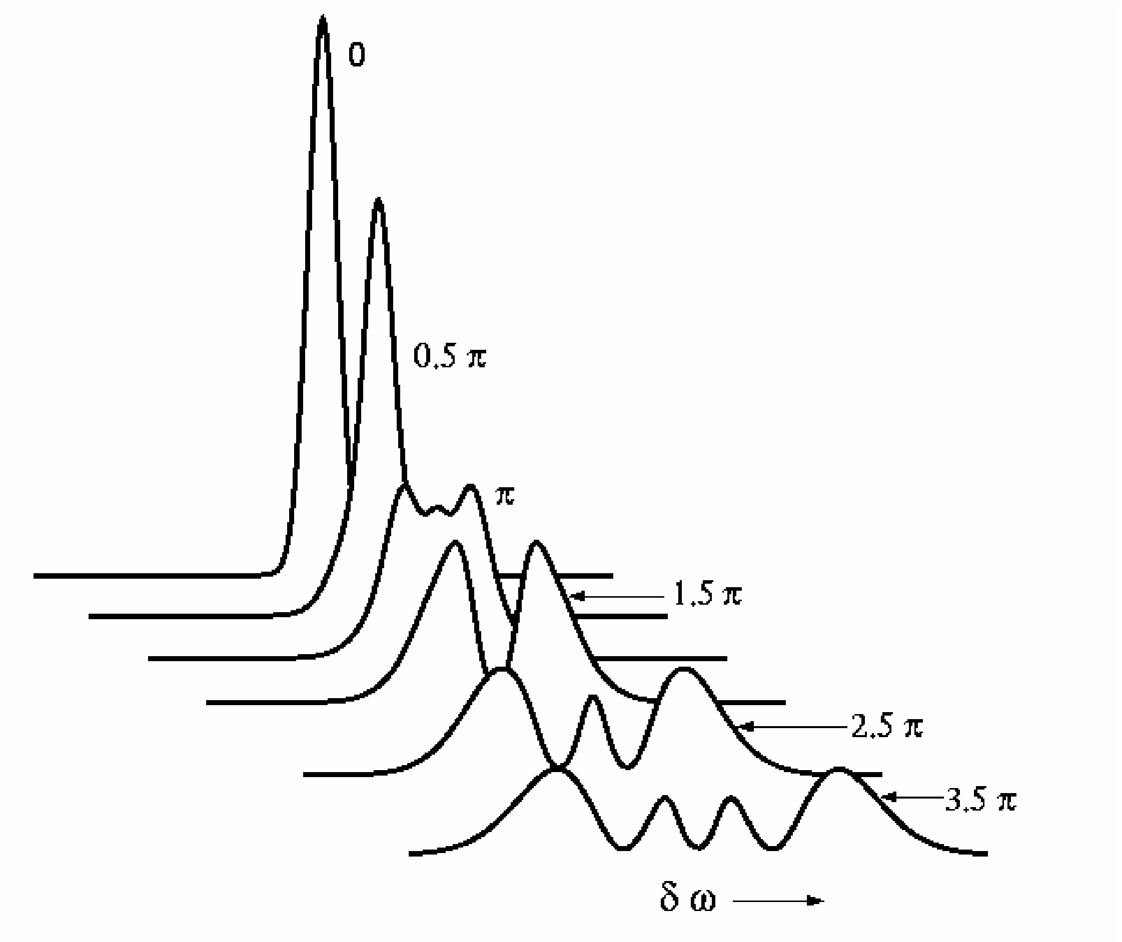
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} (1 + iz\, e^{-t^2})\, e^{i\omega t}\, dt = \tilde{u}(0,\omega) + iz\, \tilde{v}(\omega)$$

Note that once again, the new component is in quadrature with the original pulse.

Thus, the increase in net spectral width scales only as z^2.

SPECTRAL BROADENING OF A GAUSSIAN PULSE AT ZERO DISPERSION

(Peak non-linear phase shift indicated next to each spectrum.)



ORIGIN OF THE SOLITON

For the soliton, the NL and dispersive terms cancel each other's effects.

But, how can the tendancies to spectral and temporal broadening cancel one another?

Ans: There is no broadening of either kind to first order in dz!

The first order effects of both terms are just complementary phase shifts $d\phi(t)$.

Proof: If $f(z,t)$ is real, then the general eqn.

$$\frac{\partial u}{\partial z} = if(z,t)\,u$$

simply generates the phase change $d\phi(t) = f(0,t)\ dz$ in dz.

We have already seen how the NL term generates $d\phi(t) = |u(t)|^2 dz$

For the dispersive effect, write the eqn. in the form

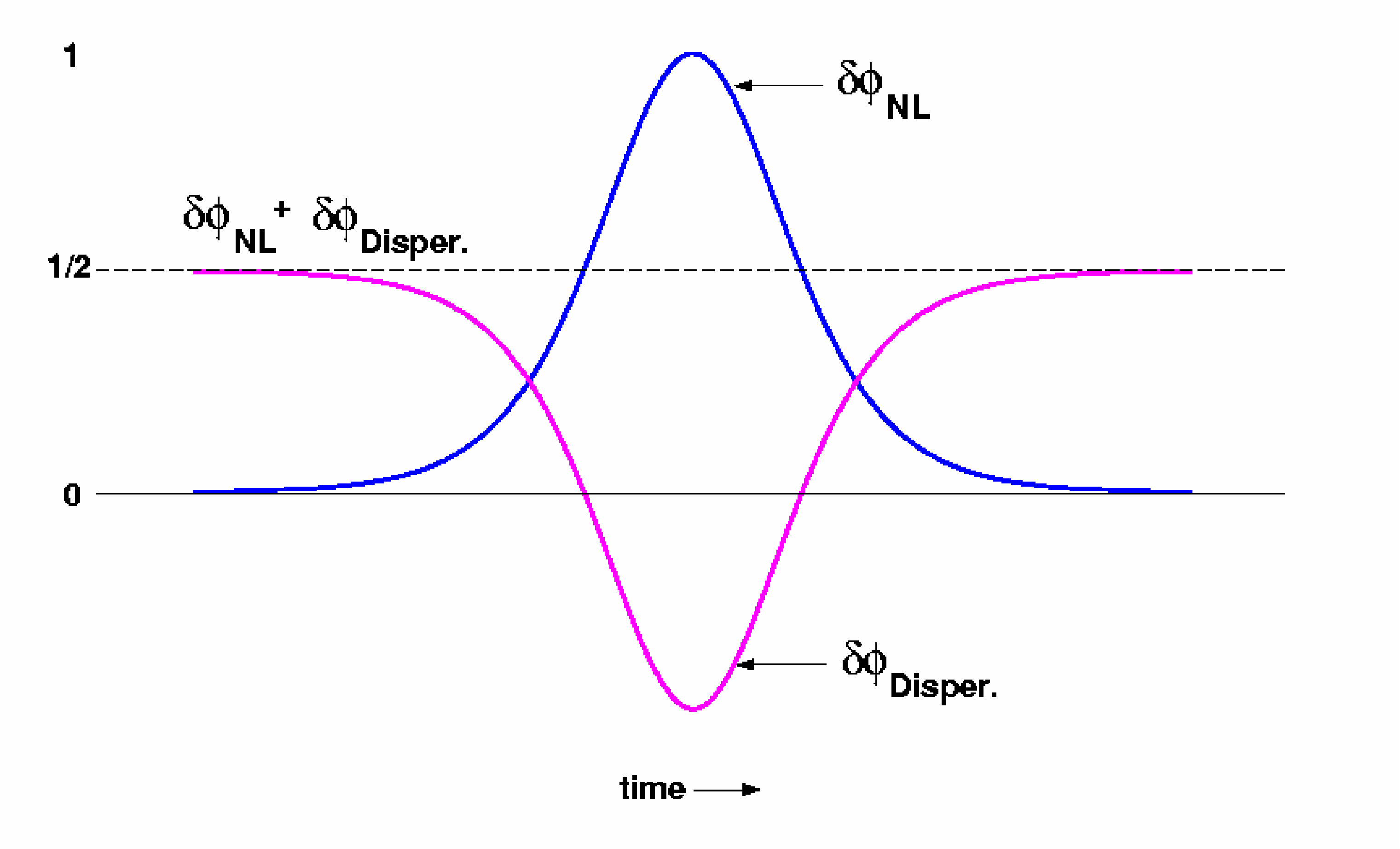
$$\frac{\partial u}{\partial z} = \left(\frac{i}{2u}\frac{\partial^2 u}{\partial t^2}\right)u$$

Thus, the dispersive term generates $d\phi = \left(\frac{1}{2u}\frac{\partial^2 u}{\partial t^2}\right)dz$

For $u = \operatorname{sech}(t)$, these terms are, respectively,

$$d\phi_{NL} = \operatorname{sech}^2(t)\ dz \quad \text{and} \quad d\phi_{disp.} = [½ - \operatorname{sech}^2(t)]\ dz.$$

THE DISPERSIVE AND NON-LINEAR PHASE SHIFTS OF A SOLITON



Note that the dispersive and non-linear phase shifts sum to a constant.

PATH-AVERAGE SOLITONS

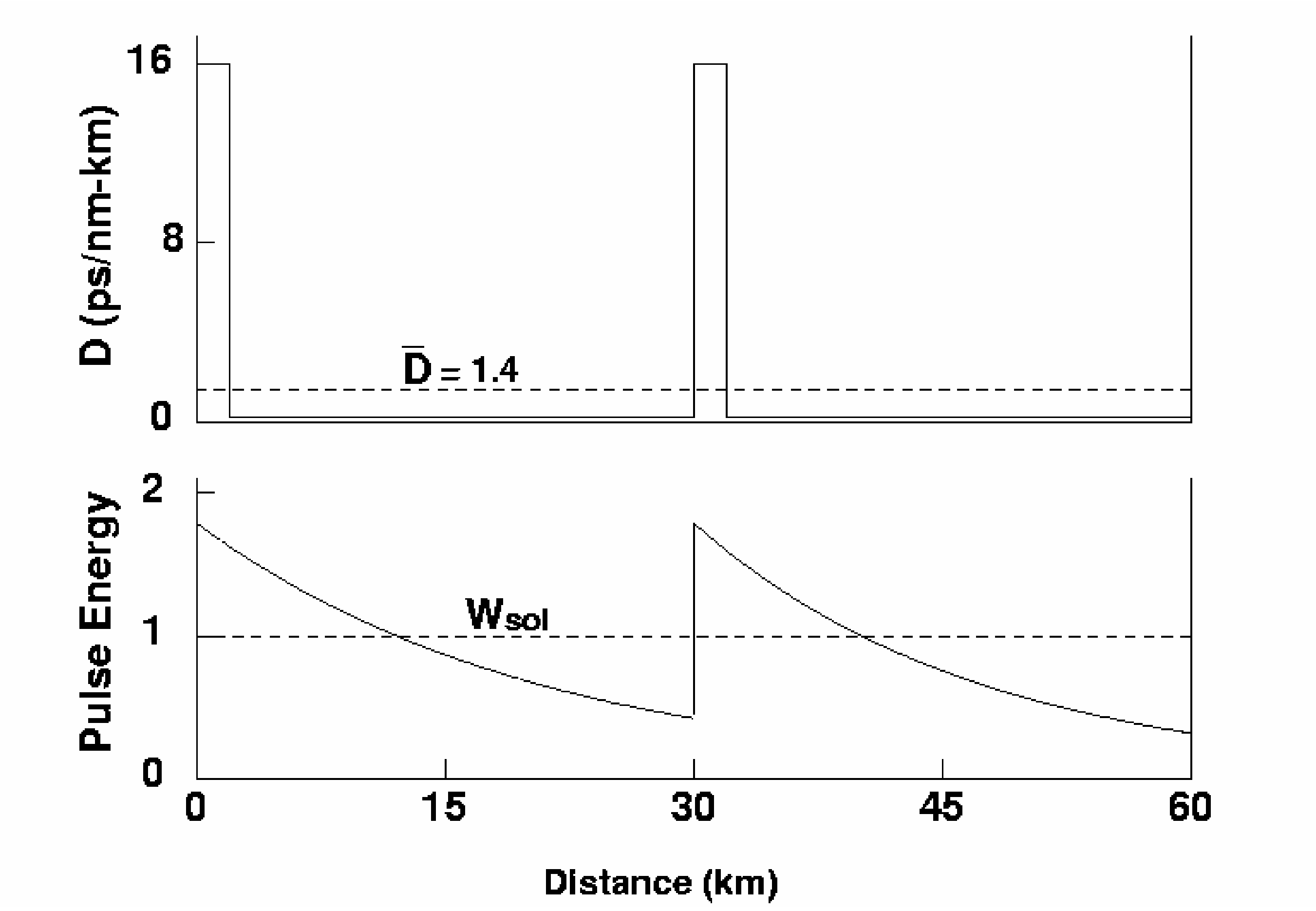
At first glance, the large periodic variations in pulse intensity in a system using lumped amplifiers spaced many tens of kilometers apart would seem to prohibit soliton propagation. (In that case, the non-linear and dispersive terms cannot cancel one another continuously, at least not when the fibers have more or less constant dispersion.)

Nevertheless, consider a system where the spacing between amplifiers, L_{amp}, is short relative to z_c, and where the non-linear and dispersive phase shifts accumulated over each L_{amp} cancel one another.

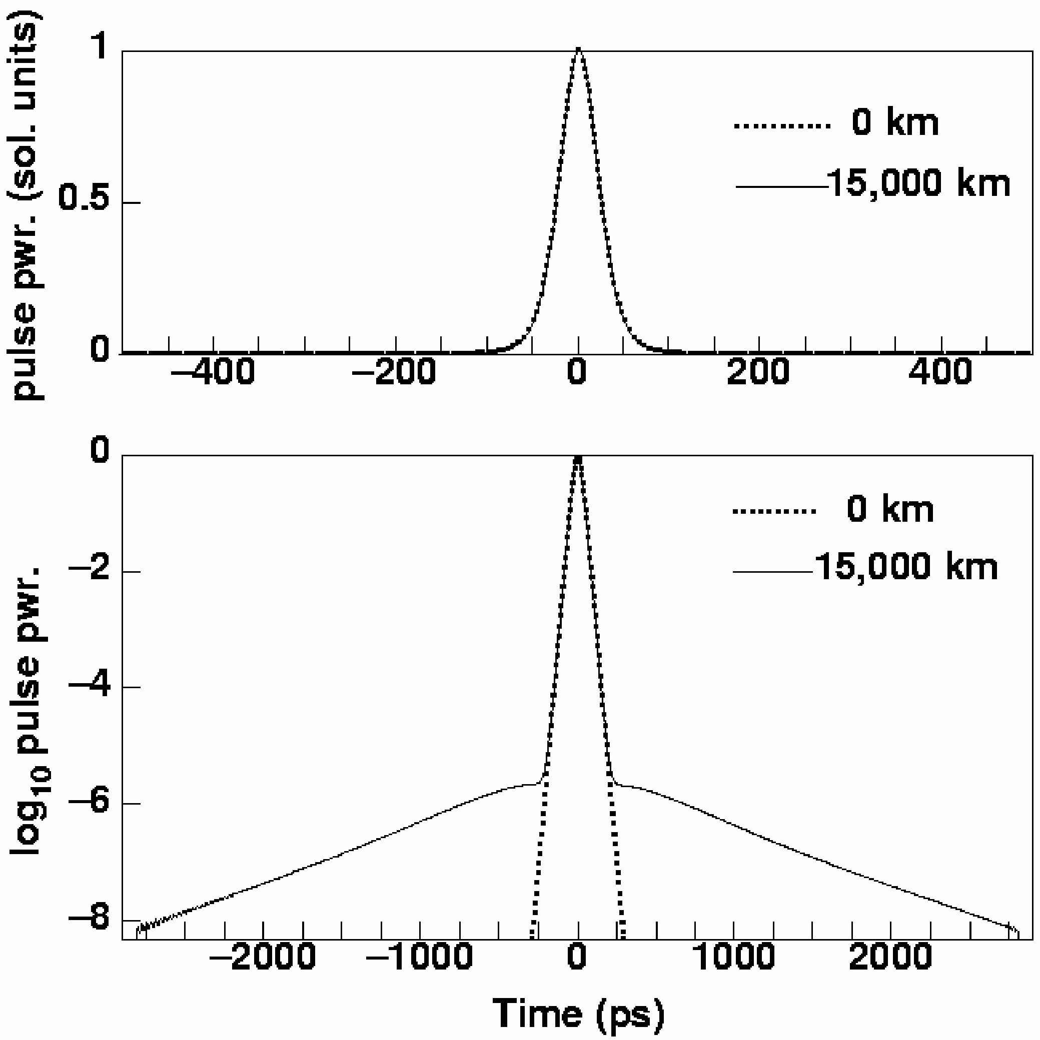
Since in that case the changes in pulse shape (both in the time and frequency domains) within each amplification period tend to be negligible, and in addition, there is no long-term accumulation of unbalanced phase shifts, one has prefectly stable pulses, known as "path-average" solitons.

This important principle works very well, as illustrated by the following example:

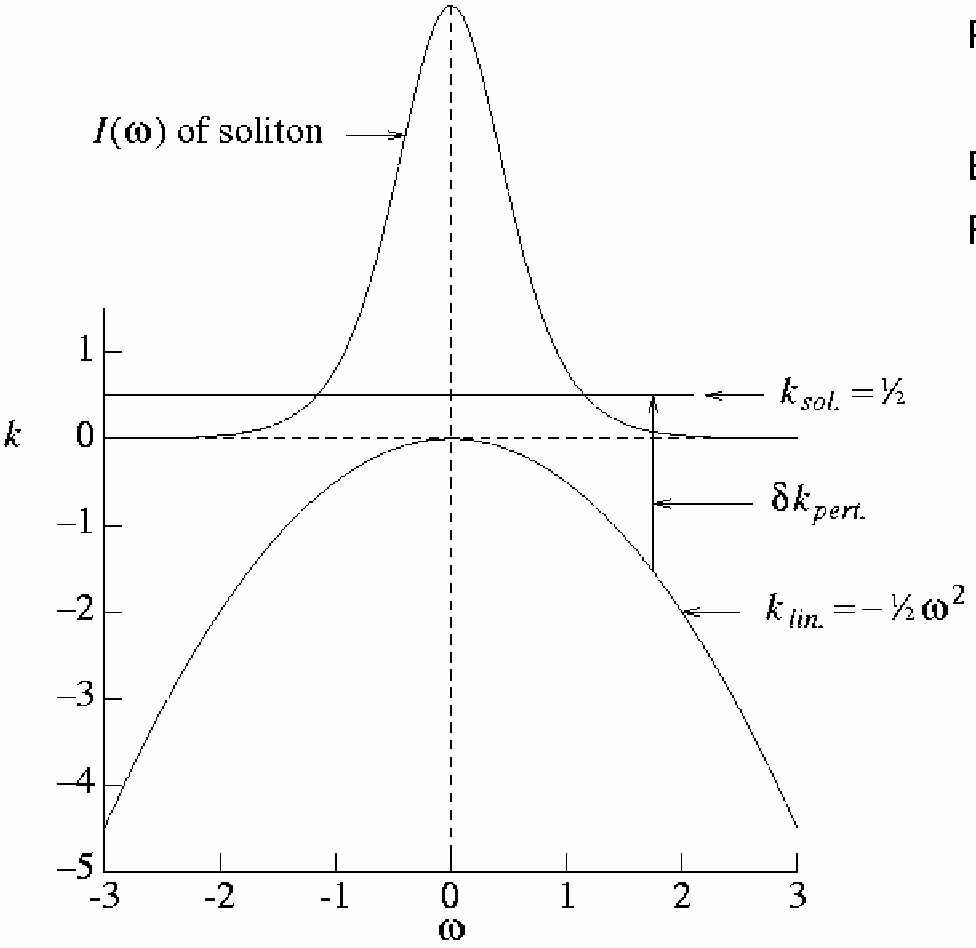
PULSE ENERGY AND FIBER DISPERSION IN SAMPLE OF TRANSMISSION LINE USED FOR TEST OF "PATH-AVERAGE" SOLITONS



SIMULATED TRANSMISSION THRU SYSTEM WITH LUMPED AMPS AND PERIODICALLY VARYING DISPERSION



DISPERSION RELATION FOR SOLITONS AND LINEAR WAVES
SOLITON SPECTRAL DENSITY ALSO SHOWN



Phase matching condition:

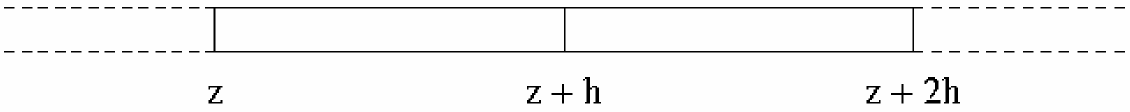
$$\delta k_{pert.} = k_{sol.} - k_{lin.}$$

Effect is $\propto I(\omega_{match})$.

Resonance $\rightarrow \delta k_{pert.} = ½$.

NUMERICAL SOLUTION OF THE NLS EQN:
THE SPLIT-STEP FOURIER TRANSFORM METHOD

The NLS equation is generally difficult to solve analytically. Numerical solution, however, can be remarkably efficient, when it is based on the "split-step Fourier" method:



The method is based on the fact that the effects of the dispersive term are most naturally dealt with in the frequency domain, while those of the non-linear term are best handled in the time domain. Thus, each increment h in z is treated in two consecutive steps, as follows:

$$u(z,t) \rightarrow \tilde{u}(z,\omega)\,; \qquad \tilde{u}(z,\omega)\; e^{-i(\omega^2/2)h} = \tilde{u}(z+h,\omega)$$

$$\tilde{u}(z+h,\omega) \rightarrow u_{new}(z,t)\,; \qquad u_{new}(z,t)\, e^{i|u|^2 h} = u(z+h,t)$$

Based on the ideas just discussed with respect to path-average solitons, reasonable accuracy can often be maintained with relatively large step sizes.

Fiber loss and amplifier gain are simulated by appropiate scale factors on $u(z)$.

Filter response functions and other frequency dependent factors are most easily applied in the frequency domain.

CONSTANTS OF THE NLS

$$-i\frac{\partial u}{\partial z} = \frac{1}{2}\frac{\partial^2 u}{\partial u^2} + |u|^2 u$$

$$W = \int |u|^2 u$$ (Pulse Energy)

$$H = \frac{1}{2}\int dt(|\frac{\partial u}{\partial t}|^2 - |u|^4)$$ (Hamiltonian)

(More)

GAUSSIAN PULSE APPROXIMATION

$$u = \sqrt{W}\left(\frac{\eta}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}(\eta + i\beta)t^2 + i\phi\right]$$

$\eta\,,\ \beta\,,\ and\ \ \phi$ **may depend on *z* but not on *t***

$$H = \frac{1}{2}W\left[\frac{\eta^2+\beta^2}{2\eta} - W\sqrt{\frac{\eta}{2\pi}}\right]$$

$$\frac{dH}{dz} = \frac{1}{4}W\left[\left(1 - \frac{\beta^2}{\eta^2} - \frac{W}{\sqrt{2\pi\eta}}\right)\frac{d\eta}{dz} + \frac{2\beta}{\eta}\frac{d\beta}{dz}\right] = 0$$

This result gives a required relation between $\frac{d\beta}{dz}$ **and** $\frac{d\eta}{dz}$

From the NLS:

$$\frac{d\eta}{dz} = 2\beta\eta$$

From the equation for H:

$$\frac{d\beta}{dz} = \beta^2 - \eta^2 + \frac{W}{\sqrt{2\pi}}\eta^{1.5}$$

These equations are very useful in picturing how solitons behave

To make a soliton: $\beta = 0$ $W = \sqrt{2\pi\eta}$