Effects of dissipative disorder on nonlinear coherent patterns: from soliton collisions in optical fibers to diverging distributions in pattern formation

Avner Peleg, The University of Arizona, Tucson, AZ

In collaboration with:
Yeo-Jin Chung (LANL) and Tomas Dohnal (UNM)

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"Dissipative disorder"

- "Dissipative disorder" – disorder that can be described as originating from randomness in the gain coefficient of a given medium.
- Why interesting? – has strong effects on the statistics of coherent localized patterns (solitons and solitary waves).

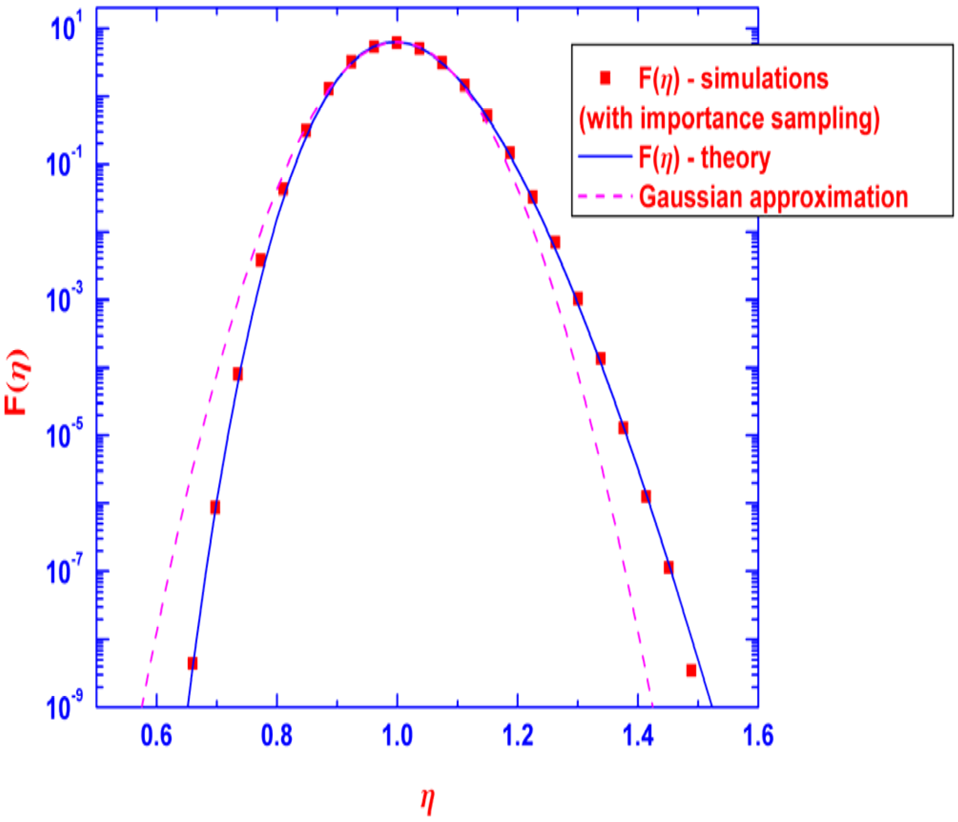
- Two examples:

Part I: Emergence of dissipative disorder in nonlinear optical fiber communication systems with multiple frequency channels **(strong effects on the statistics).**

Part II: Effects of dissipative disorder on front formation in pattern forming systems **(diverging distribution function).**

Part I: Emergence of dissipative disorder in nonlinear optical fiber communication systems with multiple frequency channels

- A. Peleg, Opt. Lett. 29, 1980 (2004).
- Y. Chung and A. Peleg, Nonlinearity, in press.

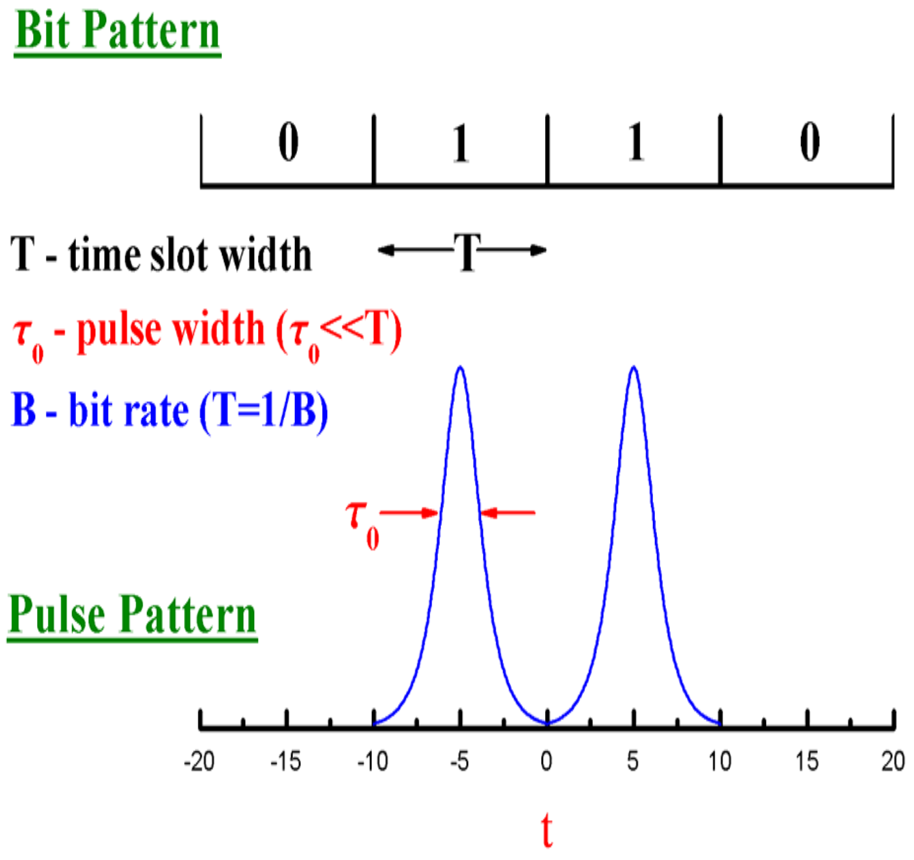


Transmission of information in fiber optics systems

- Temporal optical pulses are used as bits of information

- Each pulse is positioned at the center of a time slot allocated for the bit

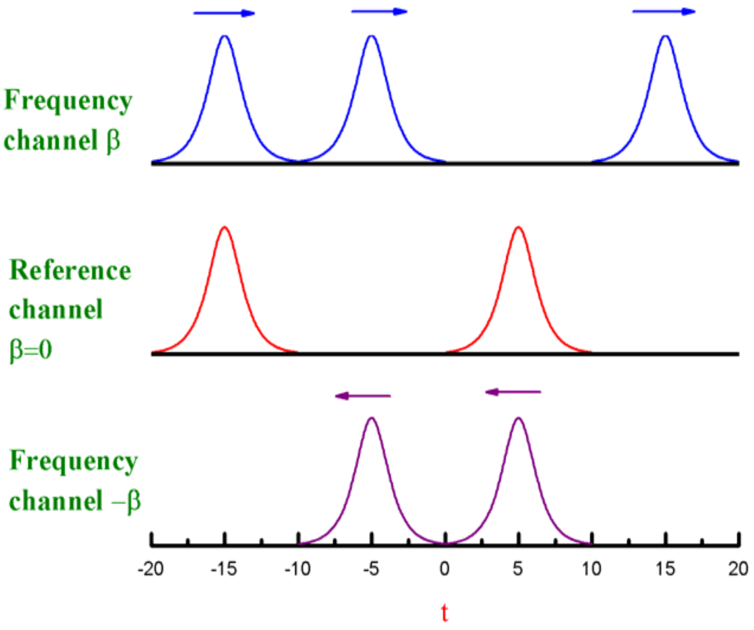
- State “1”/“0” is assigned to a slot if the slot is occupied/empty



Multichannel optical fiber systems

- Multichannel transmission: transmit **many** (more than 100) pulse sequences through the **same** fiber.

- In each frequency channel (pulse sequence) the pulses propagate with the same group velocity, but the group velocity is different for different channels.
- Collisions between pulses from different channels are very frequent.



schematic description of multichannel transmission

Effects of interchannel collisions

- A major limitation on the performance of multichannel optical fiber transmission systems is due to interchannel collisions.

- Consider conventional **optical solitons** as an example.
- In real optical fibers soliton collisions are inelastic due to perturbations, such as third order dispersion, Raman scattering, etc.
- In this case collisions might lead to: emission of radiation, change in the soliton amplitude and frequency, corruption of shape …

- Consider the effect of **delayed Raman response**, since it is expected to be dominant.

Effect of delayed Raman response on single soliton propagation

- Pulse propagation in the presence of delayed Raman response:

$$i\partial_z\Psi + \partial_t^2\Psi + 2|\Psi|^2\Psi = -\varepsilon_R \Psi \partial_t |\Psi|^2$$

Due to delayed response of the vibrational levels of SiO_2 molecules

$\varepsilon_R = 0.00594/\tau_0$ and τ_0 is the pulse width in picoseconds.

- The main effect is a downshift in the central frequency of the pulse – the Raman induced self frequency shift (Mitschke & Mollenauer 1986, Gordon 1986)

$$\frac{d\beta}{dz} = -\frac{8}{15}\varepsilon_R \eta^4$$

- It is a result of energy transfer from high frequency to low frequency components of the pulse.

NLSE with delayed Raman response

- Start with the nonlinear wave equation

$$\boxed{\nabla^2 \vec{e} - \partial_t^2 \vec{e} / c^2 = \mu_0 \partial_t^2 \vec{P}_L + \mu_0 \partial_t^2 \vec{P}_{NL}}$$

where

$$\vec{P}_{NL}(\vec{r},t) = \varepsilon_0 \iiint dt_1 dt_2 dt_3 \chi^{(3)}(t-t_1, t-t_2, t-t_3) \vdots \vec{e}(\vec{r},t_1)\vec{e}(\vec{r},t_2)\vec{e}(\vec{r},t_3)$$

and $\chi^{(3)}$ is the third order susceptibility (a 4 - th rank tensor).

- The instantaneous nonlinear response approximation (neglecting delayed Raman response):

$$\boxed{\chi^{(3)}(t-t_1, t-t_2, t-t_3) = \tilde{\chi}^{(3)} \delta(t-t_1)\delta(t-t_2)\delta(t-t_3)}$$

$$\Rightarrow \quad \boxed{\vec{P}_{NL}(\vec{r},t) = \varepsilon_0 \tilde{\chi}^{(3)} \vdots \vec{e}(\vec{r},t_1)\vec{e}(\vec{r},t_2)\vec{e}(\vec{r},t_3)}$$

$$\Rightarrow \boxed{\mathrm{n} = \mathrm{n}_0(\omega) + \mathrm{n}_2 |\mathrm{E}|^2 /2} \quad \Rightarrow \quad \boxed{\mathrm{i}\partial_z \mathrm{E} - \mathrm{d}_2 \partial_t^2 \mathrm{E} + 2\kappa |\mathrm{E}|^2 \mathrm{E} = 0} \quad \text{(ideal NLSE)}$$

NLSE with delayed Raman response

- Taking into account the delayed Raman response:

$$\boxed{\begin{array}{l}\chi^{(3)}(t-t_1,t-t_2,t-t_3)=\tilde{\chi}^{(3)}R(t-t_1)\delta(t-t_2)\delta(t-t_3)\\ \text{where } R(t-t_1)=0 \text{ for } t_1>\mathrm{t}, \text{ and } \int \mathrm{dt}R(t)=1.\end{array}}$$

- Using the slow varying envelope approximation

$$\boxed{\tilde{P}_{NL}(\vec{r},t)=\varepsilon_0\chi^{(3)}\tilde{E}(\vec{r},t)\int_{-\infty}^{t}dt_1R(t-t_1)\tilde{E}^2(\vec{r},t_1)}$$

- The equation in the time domain is

$$\mathrm{i}\partial_{\mathrm{z}}\mathrm{E}+\mathrm{ik}'\partial_{\mathrm{t}}\mathrm{E}-\frac{1}{2}\mathrm{k}''\partial_{\mathrm{t}}^2\mathrm{E}=-\kappa[1-(i/\omega_0)\partial_t][E(z,t)\int_0^{\infty}R(t')\,|\,\mathrm{E(z,t-t')}\,|^2\,\mathrm{dt'}]$$

- Expanding $|E(z,t-t')|^2$ in a Taylor series near t'=0 up to 1st order in t', and going to dimensionless quantities

$$\boxed{\mathrm{i}\partial_{\mathrm{z}}\Psi+\partial_{\mathrm{t}}^2\Psi+2|\Psi|^2\Psi=-\varepsilon_{\mathrm{R}}\Psi\partial_{\mathrm{t}}|\Psi|^2-\mathrm{i}\varepsilon_{\mathrm{s}}\partial_{\mathrm{t}}(|\Psi|^2\Psi)}$$

Raman term

self-steepening term

Effect of delayed Raman response on a single collision between two solitons

- Consider a single collision between a soliton in the reference channel ($\beta=0$) and a soliton in the β channel (Chi & Wen 1989, Malomed 1991, Kumar 1998, Lakoba and Kaup 1999).
- Assumptions: $1/|\beta| \ll 1$, $\varepsilon_R \ll 1$
- An $O(\varepsilon_R)$ change of the amplitude (Raman induced amplitude change)

$$\boxed{\Delta\eta_0 = 2\eta_0\eta_\beta \operatorname{sgn}(\beta)\varepsilon_R}$$

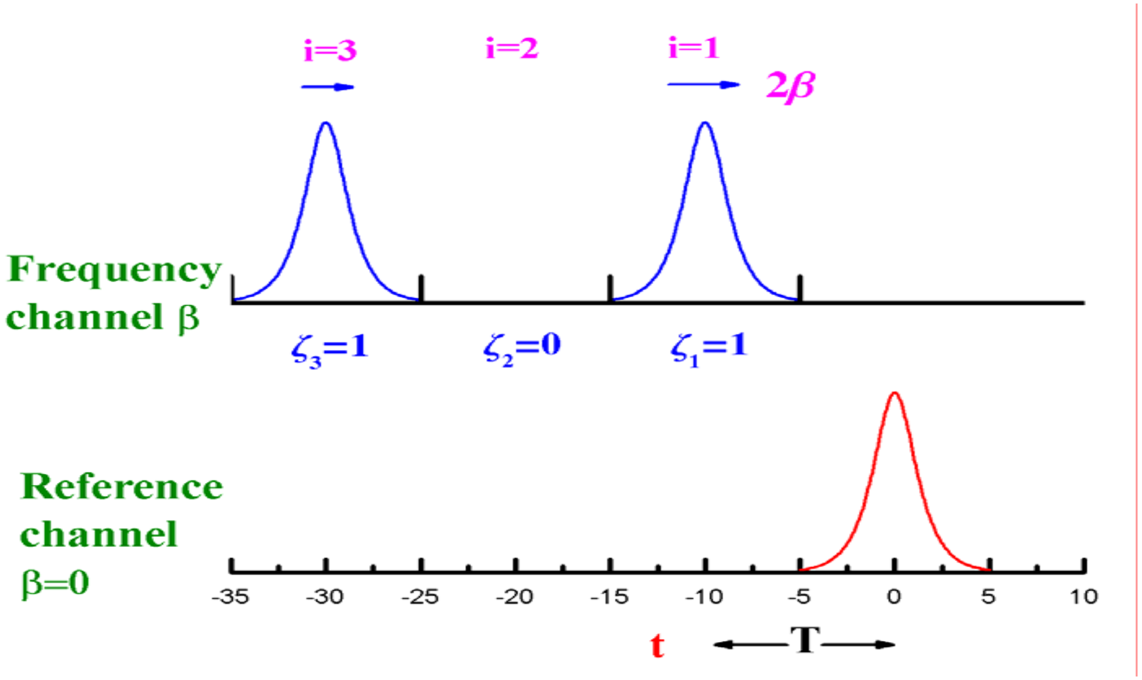
- An $O(\varepsilon_R/\beta)$ frequency change (Raman induced cross frequency shift)

$$\boxed{\Delta\beta_0 = -\frac{4\eta_0^2\eta_\beta\varepsilon_R}{|\beta|}}$$

- Assuming $\varepsilon_R \ll 1/|\beta| \ll 1$ we can neglect $O(\varepsilon_R{}^2)$ effects related to radiation emission.

Collisions with a stochastic sequence of solitons (1)

- Consider soliton propagation in the 0 channel under many collisions with a quasi-random sequence of solitons in the β channel.



- A discrete random variable ζ_i describing the occupation state of time slot i in channel β

$$\zeta_i = \begin{cases} 1 & \text{if slot i is occupied} \\ 0 & \text{if slot i is empty} \end{cases}$$

$$\langle \zeta_i \zeta_j \rangle = \begin{cases} s & \text{for } i = j \\ s^2 & \text{for } i \neq j \end{cases}$$

s - average fraction of occupied slots

Collisions with a stochastic sequence of pulses (2)

- To study fluctuations around the average cross talk effects we define a new random variable

$$\boxed{\tilde{\zeta}_i = \zeta_i - s \;\Rightarrow\; \left\langle \tilde{\zeta}_i \right\rangle = 0 \;\text{ and }\; \left\langle \tilde{\zeta}_i \tilde{\zeta}_j \right\rangle = s(1-s)\delta_{ij}}$$

- The dynamics of the amplitude at the collision point z_i is

$$\boxed{\left.\frac{\Delta\eta_0}{\Delta z_c}\right|_{z=z_i} = 2\,\mathrm{sgn}(\beta)\varepsilon_R \eta_0(z_{i-1})\frac{\tilde{\zeta}_i}{\Delta z_c} \quad \text{, where } \Delta z_c = \frac{T}{2\beta}}$$

- In the continuous limit we define

$$\boxed{\xi(z_i) = \tilde{\zeta}_i/\Delta z_c \;\Rightarrow\; \left\langle \xi(z) \right\rangle = 0 \;\text{ and }\; \left\langle \xi(z)\xi(z') \right\rangle = D\delta(z-z') \quad D = \frac{2s(1-s)\,|\beta|}{T}}$$

- The dynamics of the amplitude is described by

$$\boxed{\frac{d\eta_0}{dz} = 2\,\mathrm{sgn}(\beta)\varepsilon_R \xi(z)\eta_0(z)}$$

Collisions with a stochastic sequence of pulses (3)

$$\boxed{\eta_0(z) = \eta_0(0)\exp\left[2\,\mathrm{sgn}(\beta)\varepsilon_R \int_0^z dz'\,\xi(z')\right]}$$

- According to the central limit theorem $x(z) = \int_0^z dz'\,\xi(z')$ is a Gaussian random variable with $\langle x(z)\rangle = 0$ and $\langle x^2(z)\rangle = Dz$.

- **The distribution function of η_0 is lognormal !**

$$\boxed{F(\eta_0) = (8\pi D\varepsilon_R^2 z)^{1/2}\eta_0^{-1}\exp\left[-\frac{\ln^2(\eta_0)}{8D\varepsilon_R^2 z}\right]}$$

Implications of the lognormal amplitude distribution

- Strongly different from the Gaussian distribution.
- The moments grow exponentially with both n and z

$$\left\langle \eta_0^n \right\rangle = \exp[2n^2 D\varepsilon_R^2 z]$$

- The variance

$$\sigma_{\eta_0}^2 = \exp(4D\varepsilon_R^2 z)\left[\exp(4D\varepsilon_R^2 z) - 1\right]$$

$\Rightarrow \eta_0$ is not self - averaging

$\sigma_{\eta_0}^2 / \langle \eta_0 \rangle^2$ is not necessarily small

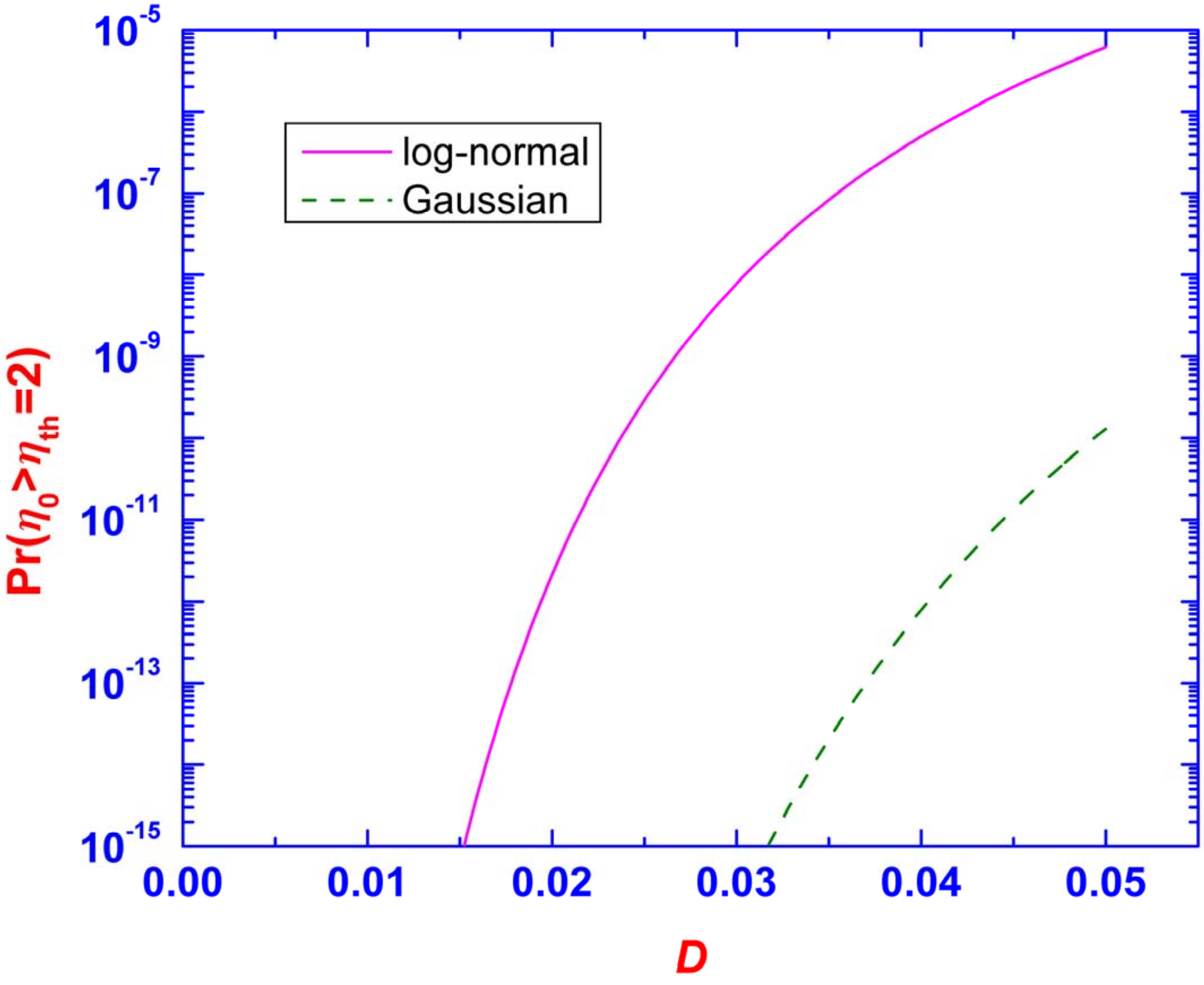
- The exponential growth of the variance with z is much stronger than the algebraic growth in the variance of the soliton parameters due to amplifier noise. (The Elgin-Gordon-Haus effect).

$\sigma_{\beta\,A}^2 \propto D_A z \quad \sigma_{\eta\,A}^2 \propto D_A z$

$\sigma_{y\,A}^2 \propto D_A z^3 \quad \sigma_{\alpha\,A}^2 \propto D_A z^3$

D_A – noise strength due to spontaneous emission in the amplfiers

Probability to observe values of η_0 that are larger than η_{th}=2



Dynamics of the soliton frequency

(1) the self frequency shift

- The Raman induced self frequency is influenced by the dynamics of η_0

$$\boxed{\frac{d\beta_0^{(s)}}{dz} = -\frac{8}{15}\varepsilon_R \eta_0^4(z)} \qquad \boxed{\beta_0^{(s)}(z) = -\frac{8}{15}\varepsilon_R \int_0^z dz' \eta_0^4(z')}$$

- Assuming that $\xi(z)$ is Gaussian

$$\boxed{\frac{\sigma^2_{\beta_0^{(s)}}}{\left\langle \beta_0^s \right\rangle^2} = \frac{1}{6}\left[\exp(32D\varepsilon_R^2 z) + 3\right]\left[\exp(32D\varepsilon_R^2 z) - 1\right]}$$

=> **$\beta_0^{(s)}$ is not self-averaging !**

Dynamics of the soliton frequency

(2) the cross frequency shift

- The dynamics of the cross frequency shift (induced by the collisions)

$$\frac{d\beta_0^{(c)}}{dz} = -\frac{8\varepsilon_R}{3|\beta|}\xi(z)\eta_0^2(z)$$

$$\boxed{\beta_0^{(c)}(z) = \frac{2}{3|\beta|}\left[1 - \exp\left(4\varepsilon_R \int_0^z dz'\,\xi(z')\right)\right]}$$

- **The distribution function of $\beta_0^{(c)}$ is also lognormal**

$$\boxed{G(\beta_0^{(c)}) = (32\pi D\varepsilon_R^2 z)^{-1/2}\left|\beta_0^{(c)} - 2/(3|\beta|)\right|^{-1}\exp\left\{-\frac{\ln^2\left[1 - 3|\beta|\beta_0^{(c)}/2\right]}{32D\varepsilon_R^2 z}\right\}}$$

Numerical simulations

- Numerically solve the equation

$$i\partial_z\Psi + \partial_t^2\Psi + 2|\Psi|^2\Psi = -\varepsilon_R\Psi\partial_t|\Psi|^2 + i\varepsilon_R \mathrm{sgn}(\beta)\xi(z)\Psi - \frac{4\varepsilon_R}{|\beta|}\xi(z)\partial_t\Psi$$

with the disorder

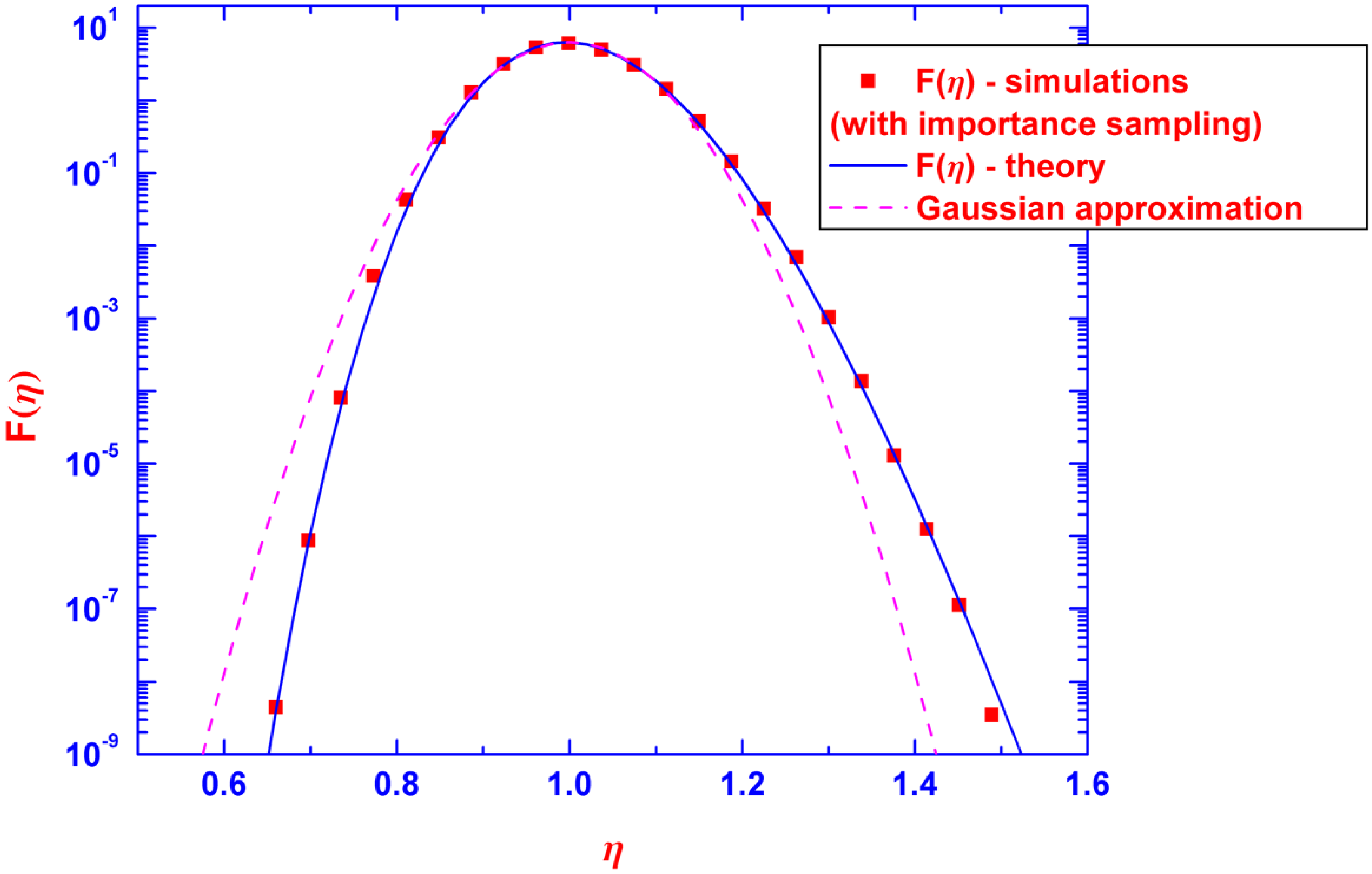
$$\langle\xi(z)\rangle = 0 \quad \text{and} \quad \langle\xi(z)\xi(z')\rangle = D\delta(z-z') \quad D = \frac{2s(1-s)|\beta|}{T}$$

and collect statistics of the amplitude η_0 and the frequency β_0.

- Values of parameters used: ε_R=0.01, β=10, D=1 (s=1/2, T=5) (Corresponding to collisions with solitons from a nearest neighbor channel).
- Tails of distribution functions were obtained by using importance sampling.

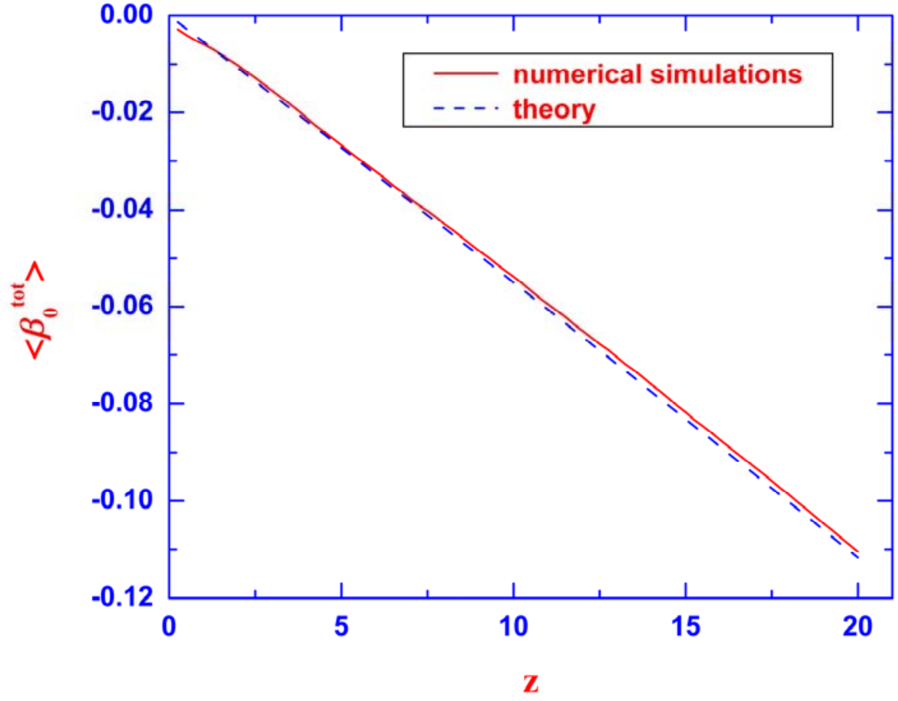
Numerical simulations

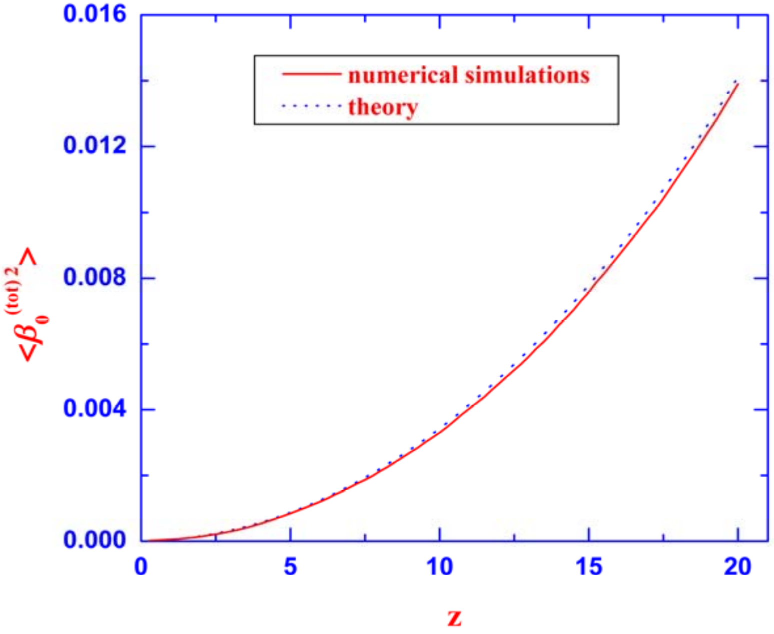
Distribution function of the amplitude F(η) at z=10



Numerical simulations

$<\beta_0^{tot}>$, $<\beta_0^{(tot)\ 2}>$, as functions of z





Numerical simulations

Distribution function of the cross frequency shift $G(\beta_0^{(c)})$ at z=10

Generalization for (2N+1) frequency channels

- The amplitude change of the solitons in a single collision is independent of the frequency difference => effect of collisions with solitons from distant channels is dominant.
- The discrete random variable describing the occupation state of the m-th slot in the k-th channel

$$\tilde{\zeta}_{mk} = \zeta_{mk} - \mathrm{s} \;\Rightarrow\; \left\langle \tilde{\zeta}_{mk} \right\rangle = 0 \quad \text{and} \quad \left\langle \tilde{\zeta}_{mk} \tilde{\zeta}_{nl} \right\rangle = \mathrm{s}(1-\mathrm{s})\delta_{\mathrm{mn}}\delta_{\mathrm{kl}}$$

- The dynamics of η_0 is approximately described by

$$\left.\frac{\Delta\eta_0}{\Delta z_c^{(1)}}\right|_{z=z_i} = \frac{2\varepsilon_R \eta_0(z_{i-1})}{\Delta z_c^{(1)}} \sum_{k\neq 0} \mathrm{sgn}(\beta_k) \sum_{m=k(i-1)+1}^{ki} \tilde{\zeta}_{mk} \qquad \Delta z_c^{(1)} = \frac{T}{2\beta^{(1)}}$$

$\beta^{(1)}$ – the frequency difference between neighboring channels

- The continuous disorder field for the 2N+1 channels case

$$\left\langle \xi_N(z)\xi_N(z')\right\rangle = D_N\delta(z-z') \quad D_N = N(N+1)D$$

- As a result

$$\frac{d\eta_0}{dz} = 2\varepsilon_R\xi_N(z)\eta_0(z)$$

and

$$\boxed{F(\eta_0) = (8\pi D_N\varepsilon_R^2 z)^{1/2}\eta_0^{-1}\exp\left[-\frac{\ln^2(\eta_0)}{8D_N\varepsilon_R^2 z}\right]}$$

similar to the 2 channels case with D_N replacing D.

- For the self frequency shift

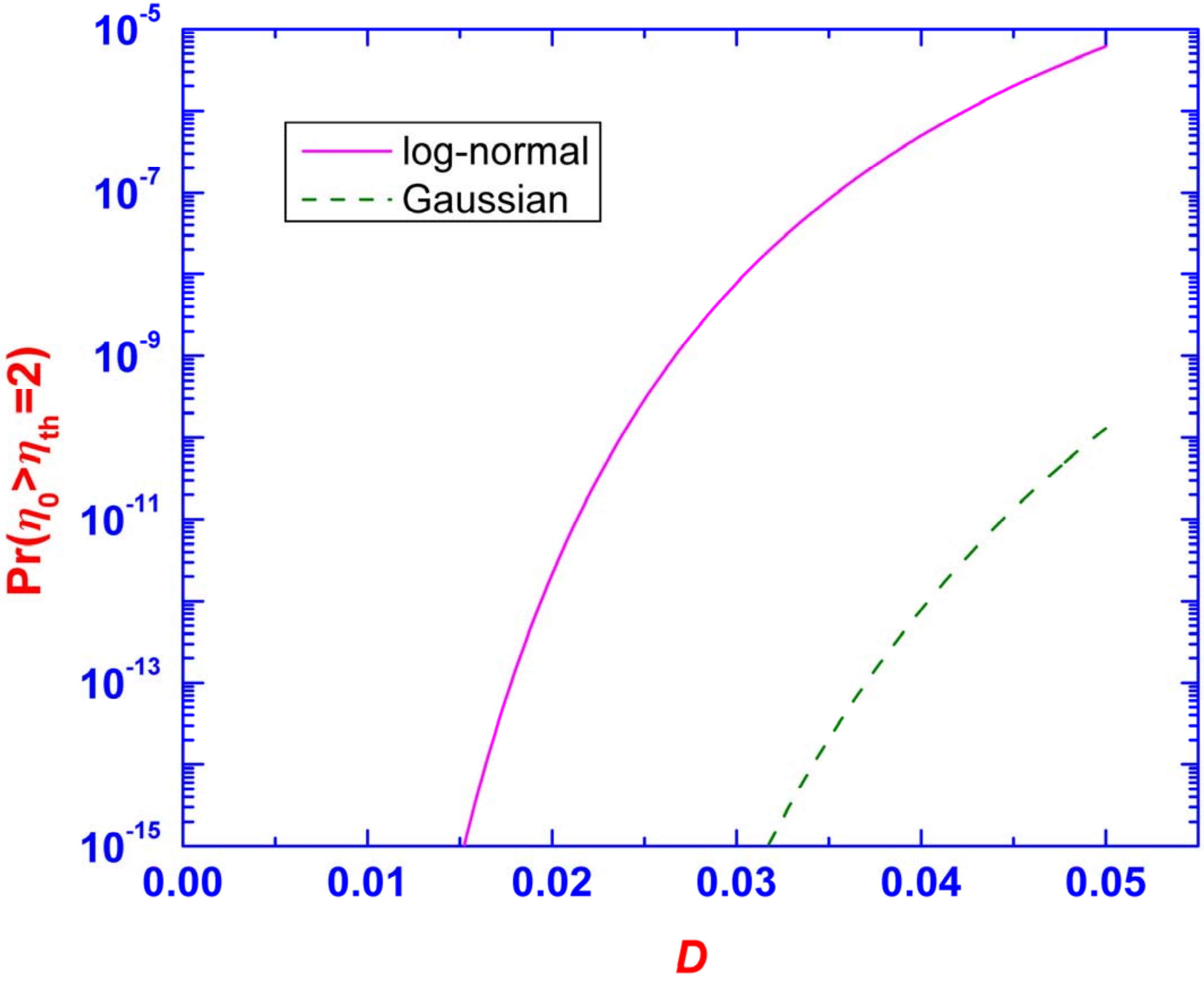
$$\boxed{\frac{\sigma^2_{\beta_0^{(s)}}}{\left\langle \beta_0^s\right\rangle^2} = \frac{1}{6}\left[\exp(32D_N\varepsilon_R^2 z)+3\right]\left[\exp(32D_N\varepsilon_R^2 z)-1\right]}$$

Example: a multichannel system operating at 10Gbit/s

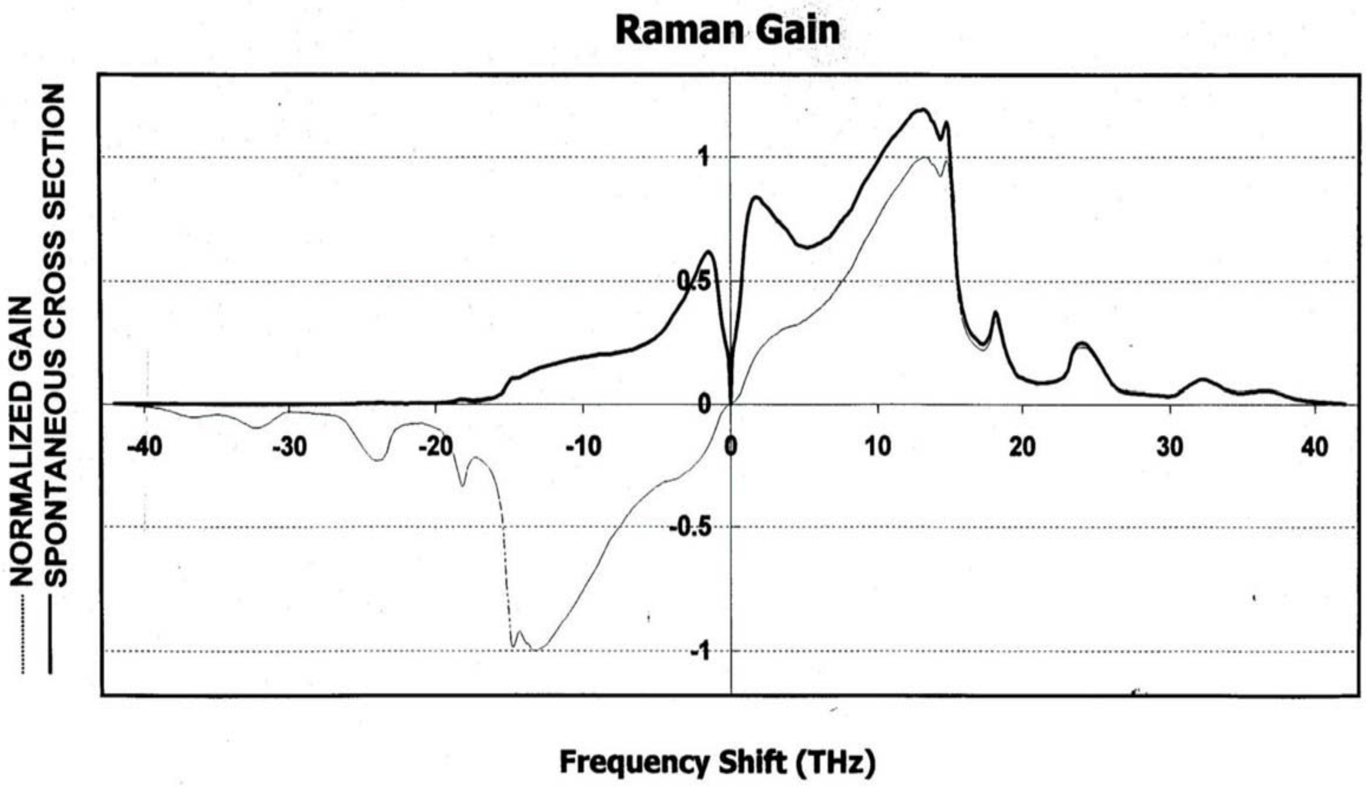
- Pulse width τ_0=20ps, $\varepsilon_R=0.00594/\tau_0=\mathbf{3x10^{-4}}$
- Channel spacing $\beta^{(1)}=10$ (corresponding to $\Delta\lambda$=0.6nm $\Delta\nu$=75GHz)
- Assuming T=5 and s=1/2, D=1.
- The peak of the Raman gain curve is at 13.2THz => the number of channels satisfying the Raman gain triangular approximation is 175.
- 2N+1=101 channels, λ=1.55μm (minimum fiber loss), β_2=-1ps^2/km (second order dispersion coefficient).
- Collision length = 40km, "Intercollision distance"= $\Delta z_c^{(1)}$= 200km

- The disorder strength for this system is: $32N(N+1)D\varepsilon_R^2 z=7.34x10^{-3}z$
- For z=6.25 corresponding to 5000km: $32N(N+1)D\varepsilon_R^2 z|_{z=6.25}=\mathbf{0.05}$

Probability to observe values of η_0 that are larger than η_{th}=2



The Raman gain curve



Normalized Raman gain as a function of frequency separation (in THz). The maximum is at 13.2 THz. (From R.H. Stolen, “Fundamentals of Raman amplification in fibers”, in “Raman amplifiers for telecommunications”, edited by M.N. Islam).

Comparison with other sources of noise and disorder

noise/disorder	**affected parameters**	**Statistics**	**variance**	**References**
amplifier noise (Gordon-Haus effect)	η, β, y, α	"Gaussian-like"	η, β $D_A z$ y, α $D_A z^3$	Gordon&Haus 1986
birefringence disorder (deviations from cylindrical shape)	y	Gaussian	y $D_B z$ (1 pulse) y $D_B^2 z^3$ (2 pulses)	Lakoba&Kaup 1997 Chung et al. 2004
disorder in dispersion coefficient	y	Gaussian	y $D_D^2 z^3$ (2 pulses)	Chertkov et al. 2003
amplifier noise + in-line phase modulation control	η, β, y, α	non-Gaussian		Falkovich et al. 2001, 2004
Raman induced energy exchange	η, β, y, α	lognormal-like	exponential in z	Peleg 2004 Chung&Peleg, 2005 (in press)

conclusions, Part I

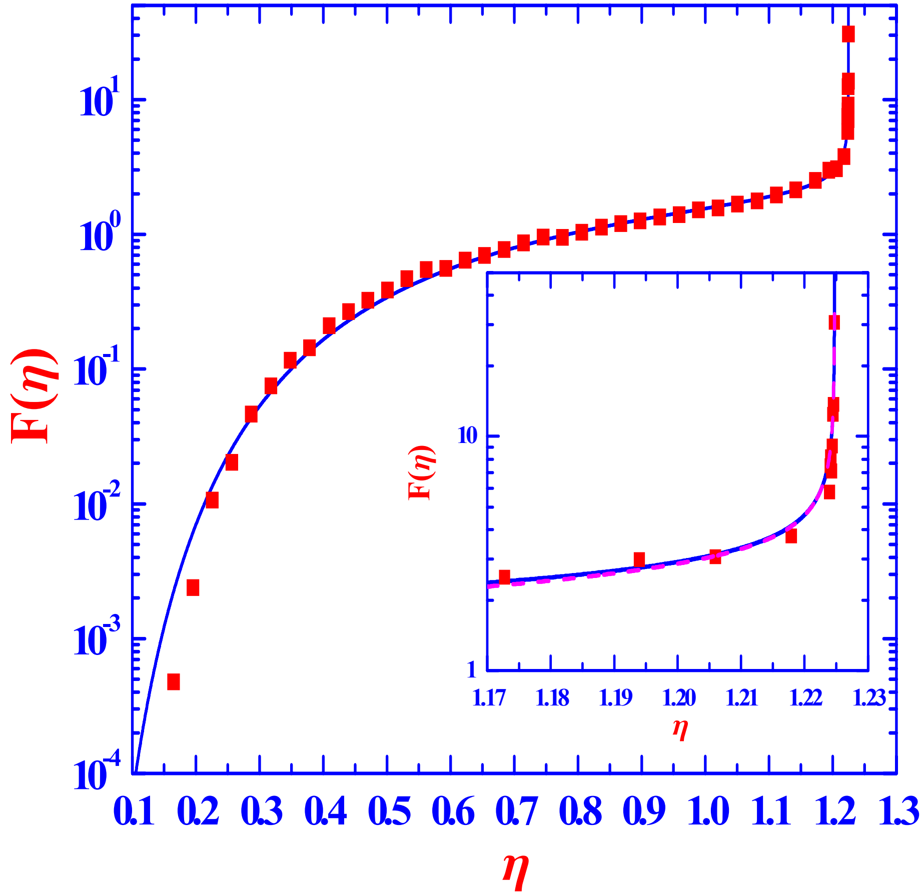
- Raman induced cross talk + stochastic nature of pulse sequences => lognormal distribution for the pulse amplitude.
- The cross frequency shift is also lognormally distributed, and the self frequency shift is not self-averaging.
- **Fluctuations in amplitude and frequency play an important role in multichannel soliton transmission.**

- The disorder strength is proportional to the square of the number of channels in the system.

- Future work: (1) Dispersion managed fibers; (2) Raman amplification in multichannel transmission.

Part II: Effects of weak dissipative disorder on front formation in pattern forming systems

- **A. Peleg, T. Dohnal, and Y. Chung, submitted.**



Introduction

- Most pattern forming systems in nature evolve under the influence of noise/disorder.

- Strong disorder => patterns are usually destroyed.
- Weak disorder => patterns can from and evolve.

- What are the statistics of the parameters characterizing the emerging patterns?

- Fronts are among the most common patterns => study front formation induced by noise/disorder.

Cubic-quintic nonlinear Schrödinger equation (CQNLSE) with disorder in the linear gain

$$\mathrm{i}\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = \mathrm{i}\varepsilon\xi(\mathrm{z})\psi$$

$$\langle\xi(\mathrm{z})\rangle = 0, \qquad \langle\xi(\mathrm{z})\xi(\mathrm{z}')\rangle = D\delta(z-z')$$

Nonlinear optics notation
ψ – envelope of electric field,
z – propagation distance, t – time,
ε_q – quintic nonlinearity coefficient, $0<\varepsilon<<1$ – linear gain coefficient
D – disorder intensity

- CQNLSE and its extension the complex cubic-quintic Ginzburg-Landau equation are used to model a wide variety of physical systems:

Pulse propagation in optical waveguides (Mihalache et al. 1988),
Mode-locked lasers (Moores 1993),
Convection in fluids (Deissler and Brand 1994),
Plasma-laser interaction (Liu et al. 2004).

Why study disorder in the linear gain coefficient?

- Such disorder is quite common in optical fiber and fiber laser systems due to fluctuations in the linear gain/loss coefficient.

- Effective disorder in the linear gain coefficient plays an important role in optical fiber communication systems with multiple frequency channels (part I of the talk).

- A *deterministic* linear gain/loss term is included as part of the cubic-quintic complex Ginzburg-Landau equation.

- The solitary waves of the CQNLSE become unstable in the presence of linear gain and evolve into fronts.

Solitary waves of the CQNLSE

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = i\varepsilon\xi(z)\psi$$

$$\langle\xi(z)\rangle = 0, \qquad \langle\xi(z)\xi(z')\rangle = D\delta(z-z')$$

- When ε=0, the equation has solitary wave solutions of the form

$$\psi_s(t,z) = \Psi_s(x)\exp(i\chi)$$

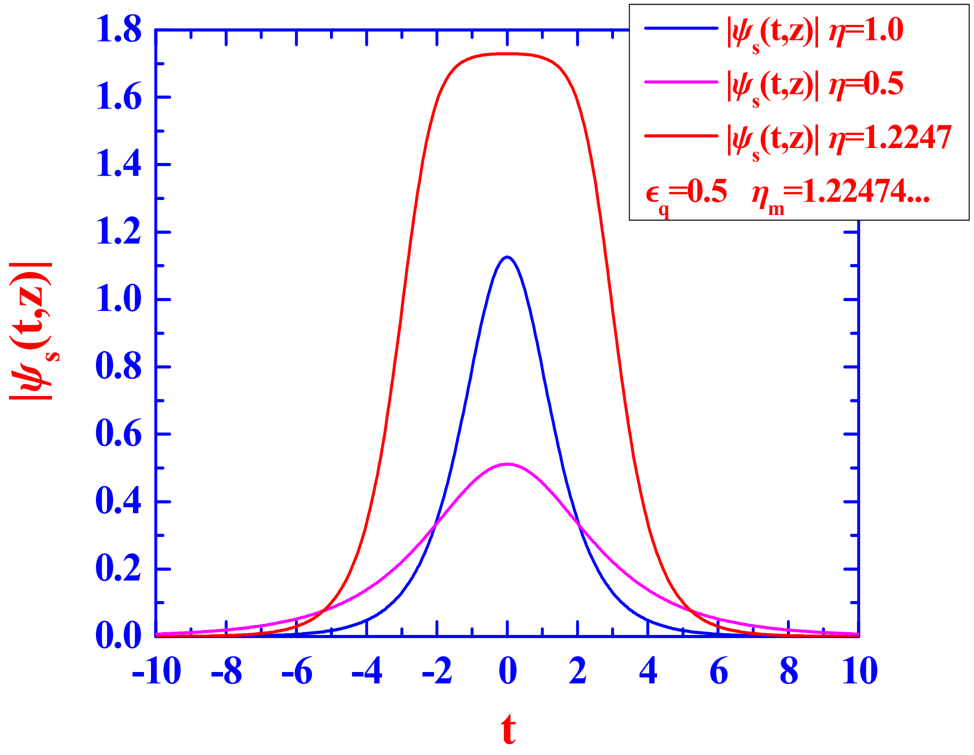
where

$$\Psi_s(x) = \sqrt{2}\eta[(1-4\varepsilon_q\eta^2/3)^{1/2}\cosh(2x)+1]^{-1/2}$$

$$x = \eta(t-y-2\beta z)$$

$$\chi = \alpha + \beta(t-y) + (\eta^2-\beta^2)z$$

- η, β, α, and y are related to the amplitude, group velocity, phase and position of the solitary wave.
- These solutions exist provided $\eta < \eta_m \equiv (4\varepsilon_q/3)^{-1/2}$.



Pulse dynamics in the presence of weak disorder in the linear gain coefficient

- Assumptions: $\varepsilon_q > 0$ (front formation); $4D\varepsilon^2 z \ll 1$ (first order adiabatic perturbation theory).

- The dynamics of the solitary wave amplitude is determined by

$$\boxed{\partial_z \int_{-\infty}^{\infty} dt \left|\psi(t,z)\right|^2 = 2\varepsilon\xi(z)\int_{-\infty}^{\infty} dt \left|\psi(t,z)\right|^2}$$

- Using the adiabatic perturbation theory we obtain

$$\boxed{\frac{d}{dz}\ln\left[\operatorname{arctanh}\left(\frac{\eta(z)}{\eta_m}\right)\right] = 2\varepsilon\xi(z)}$$

Integrating over z

$$\boxed{\begin{array}{l} \eta(z) = \eta_m \tanh\left\{\mathrm{c}(0)\exp[2\varepsilon\int_0^z dz'\,\xi(z')]\right\} \\ \text{where } \mathrm{c}(0) = \operatorname{arctanh}[\eta(0)/\eta_\mathrm{m}] \end{array}}$$

Amplitude distribution function

$$\boxed{\eta(z) = \eta_m \tanh\left\{ \mathrm{c}(0) \exp[2\varepsilon \int_0^z dz' \xi(z')] \right\}}$$

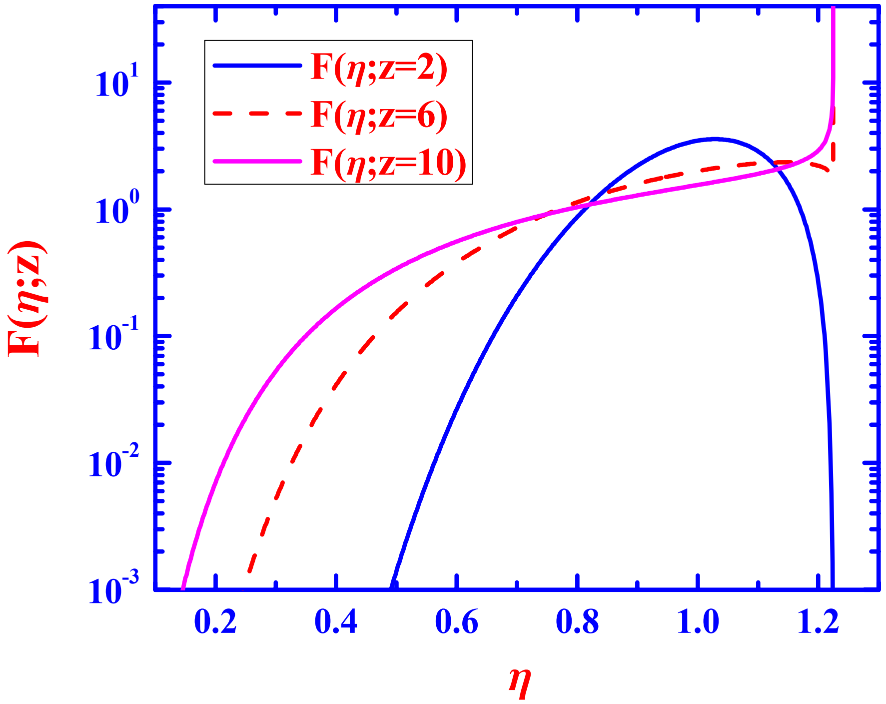
- ξ(z) is short correlated => $x(z) = \int_0^z dz' \xi(z')$ is a Gaussian random variable with $\langle \mathrm{x(z)} \rangle = 0$ and $\langle \mathrm{x}^2\mathrm{(z)} \rangle = \mathrm{Dz}$.

- The distribution function of η is

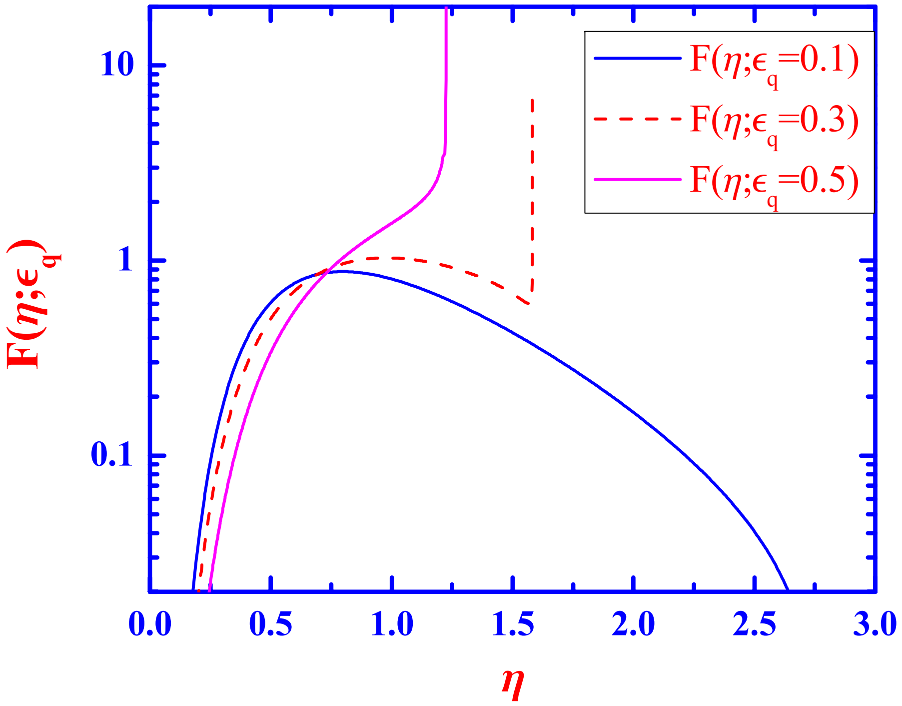
$$\boxed{F(\eta) = \begin{cases} \dfrac{\exp\left\{-\ln^2\left[\mathrm{arctanh}[\eta / \eta_m] / \mathrm{c}(0)\right] / (8D\varepsilon^2 z)\right\}}{(8\pi D\varepsilon^2 z)^{1/2} \eta_m (1 - \eta^2 / \eta_m^2) \mathrm{arctanh}(\eta / \eta_\mathrm{m})} & 0 \le \eta < \eta_m \\ 0 & \text{elsewhere} \end{cases}}$$

Dynamics of the amplitude distribution $F(\eta)$

$F(\eta;z)$ at different z
$\varepsilon_q=0.5$, $\varepsilon=0.05$, $D=3$, $\eta(0)=1$



$F(\eta;\varepsilon_q)$ at different ε_q
$z=10$, $\varepsilon=0.05$, $D=3$, $\eta(0)=1$



Properties of the amplitude distribution F(η)

- **Near η_m F(η) has a loglognormal diverging form**
- Denote $\delta\eta = \eta_m - \eta$ where $0 < \delta\eta/\eta_m \ll 1$

$$\boxed{x(z) \approx \ln\left[-\ln[\delta\eta/(2\eta_m)]/(2\mathrm{c}(0))\right]/(2\varepsilon)}$$

$$\Rightarrow \boxed{F(\delta\eta) \approx \frac{\exp\left\{-\ln^2\left[-\ln[\delta\eta/(2\eta_m)]/(2\mathrm{c}(0))\right]/(8D\varepsilon^2 z)\right\}}{(8\pi D\varepsilon^2 z)^{1/2}\,\delta\eta\,|\ln(\delta\eta/2\eta_\mathrm{m})|}}$$

- When $\varepsilon_q \ll \varepsilon \ll 1$ (cubic NLSE case) F(η) approaches the **lognormal** distribution

$$\boxed{x(z) \approx \ln[\eta/\eta(0)]/(2\varepsilon)}$$

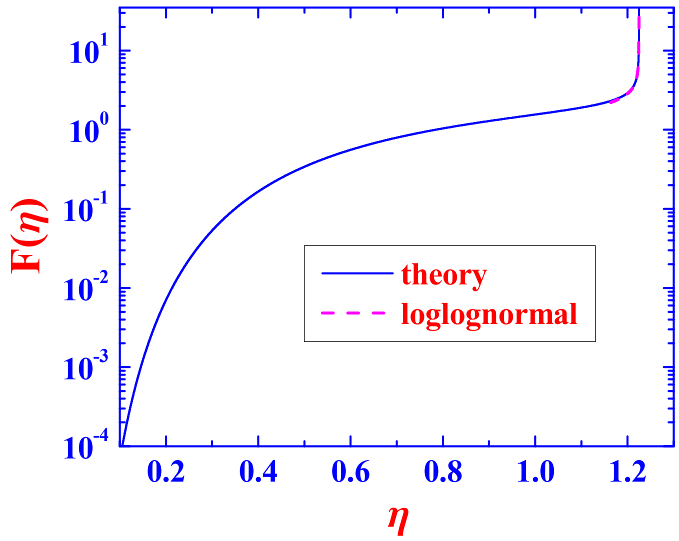
$$\Rightarrow \boxed{F(\eta) \approx \frac{\exp\left[-\ln^2(\eta/\eta(0))/(8D\varepsilon^2 z)\right]}{(8\pi D\varepsilon^2 z)^{1/2}\,\eta}}$$

Loglognormal divergence vs. lognormal statistics

Strongly nonintegrable limit

$\boldsymbol{\varepsilon \ll \varepsilon_q}$

F(η) has a compact support and a **loglognormal diverging** form near η_m

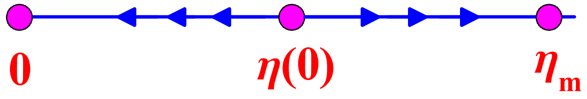


pulse decay

$\xi(z)<0$

front formation

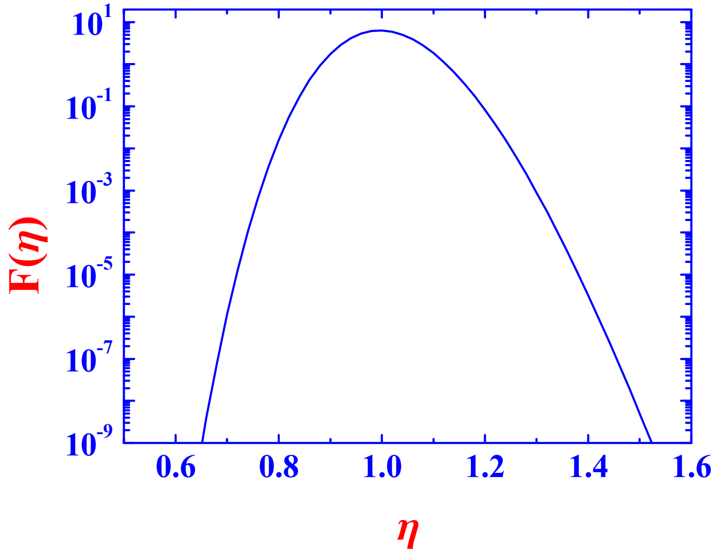
$\xi(z)>0$



Integrable (cubic NLS) limit

$\boldsymbol{\varepsilon_q \ll \varepsilon \ll 1}$

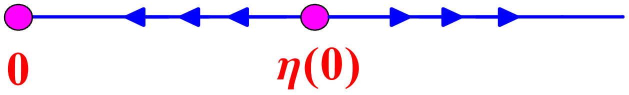
$\eta_m \rightarrow \infty$ and F(η) approaches a **lognormal distribution**



pulse decay

$\xi(z)<0$

$\xi(z)>0$



Statistics of the front position t_{fr}

- Define $t_{fr}(z)$ by $|\psi(t_{fr},z)| \equiv (1/2)\max|\psi(t,z)|$

$$\boxed{t_{fr}(z) \equiv \operatorname{arccosh}\left[4+3(1-\eta^2/\eta_{\mathrm{m}}^2)^{-1/2}\right]/(2\eta)}$$

- The field $\psi(t,z)$ corresponds to a front if $B\eta_m < \eta(z) < \eta_m$, where we choose B=0.95.

- The distribution of t_{fr}

$$\boxed{G(t_{fr}) = \left[\int_{B\eta_m}^{\eta_m} F(\eta)d\eta\right]^{-1}\left(\frac{dt_{fr}}{d\eta}\right)^{-1} F(\eta(t_{fr}))}$$

The tail of $G(t_{fr})$ is lognormal

- Consider

$$\partial_z \int_{-\infty}^{\infty} dt \left|\psi(t,z)\right|^2 = 2\varepsilon\xi(z)\int_{-\infty}^{\infty} dt \left|\psi(t,z)\right|^2$$

- In the limit $\eta \rightarrow \eta_m$

$$\int_{-\infty}^{\infty} dt\,|\psi|^2 \approx b\,\eta_m t_{fr} \;\Rightarrow\; dt_{fr}/dz \approx 2\varepsilon\xi(z)t_{fr}$$
$$\Rightarrow\; t_{fr}(z) \approx \mathrm{const}\times\exp[2\varepsilon \mathrm{x}(\mathrm{z})]$$

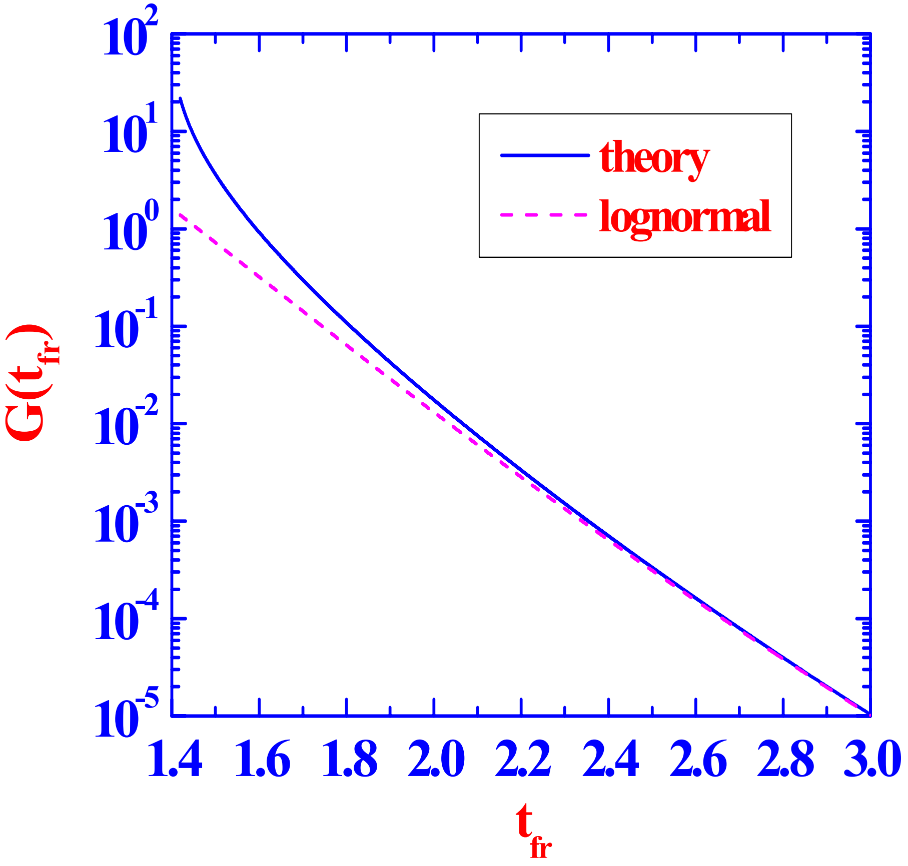
$\Rightarrow$ t_{fr} is lognormally distributed.

- A more detailed calculation leads to

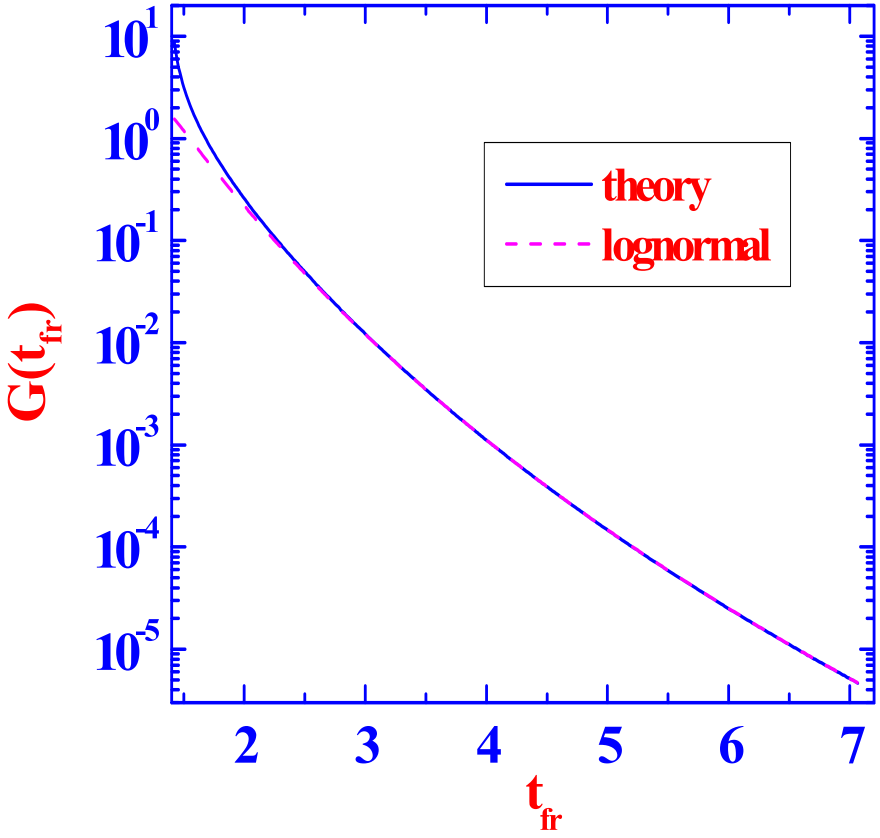
$$G(t_{fr}) \approx \frac{C\eta_m \exp\left\{-\ln^2\left[(2\eta_m t_{fr} - \ln 3)/\mathrm{c}(0)\right]/(8D\varepsilon^2 z)\right\}}{(2\pi D\varepsilon^2 z)^{1/2}(2\eta_m t_{fr} - \ln 3)}$$

$G(t_{fr})$

$\varepsilon_q=0.5$, $\varepsilon=0.03$, $z=10$, $D=3$, $\eta(0)=1$



$\varepsilon_q=0.5$, $\varepsilon=0.05$, $z=10$, $D=3$, $\eta(0)=1$



Numerical simulations

- Numerically solve

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = i\varepsilon\xi(z)\psi$$
$$\langle\xi(z)\rangle = 0, \qquad \langle\xi(z)\xi(z')\rangle = D\delta(z-z')$$

and collect statistics of η and t_{fr}.

- Two sets of parameters:

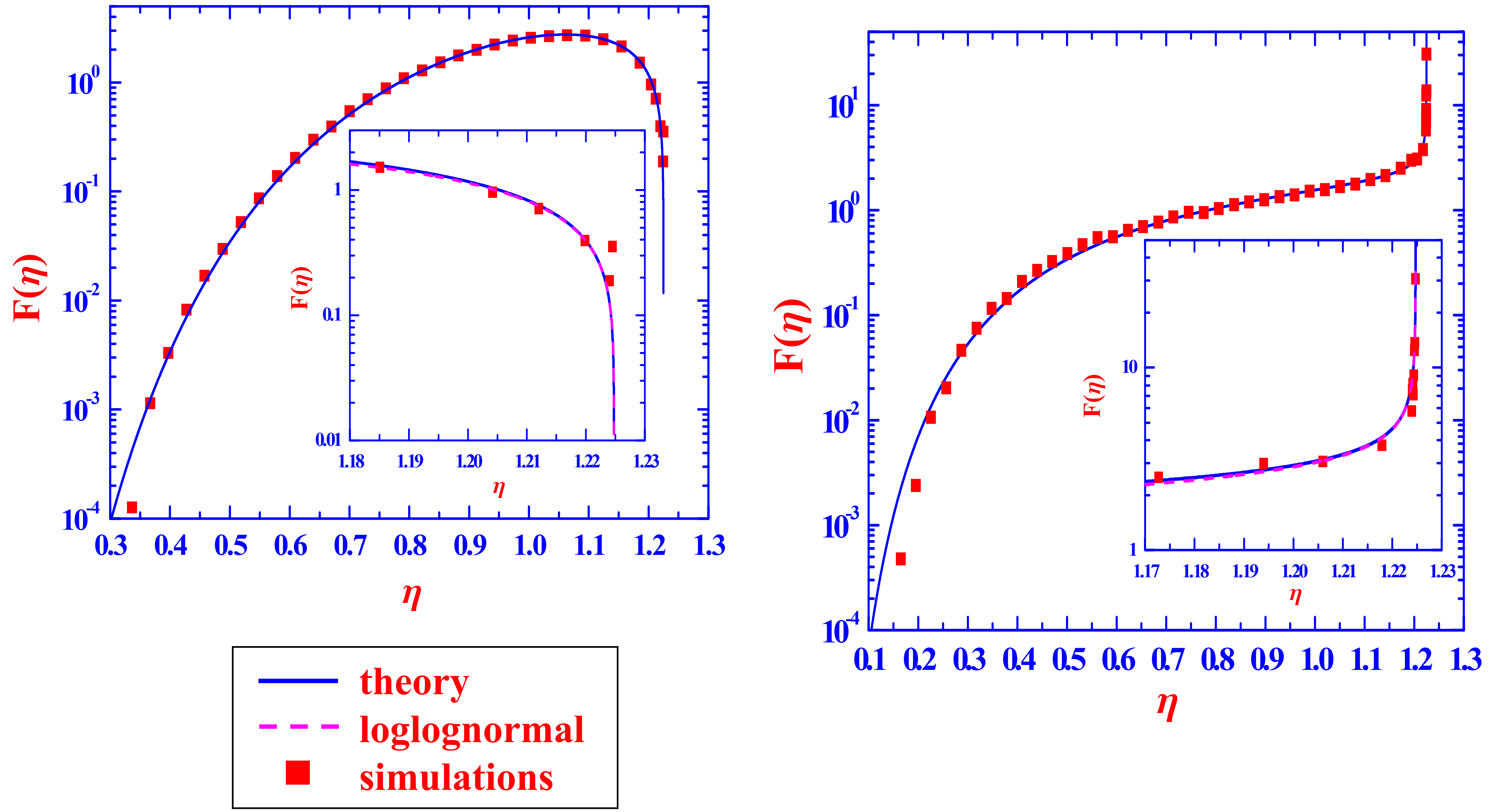
(1) ε_q=0.5, ε=0.03, D=3, z_f=10. For these values: $4D\varepsilon^2 z_f$=0.108 => **weak disorder**.

(2) ε_q=0.5, ε=0.05, D=3, z_f=10. For these values: $4D\varepsilon^2 z_f$=0.3 => **weak to intermediate disorder**.

- For both sets we sampled more than $2.5\text{x}10^5$ disorder realizations.

Numerical simulations

Amplitude distribution function F(η)



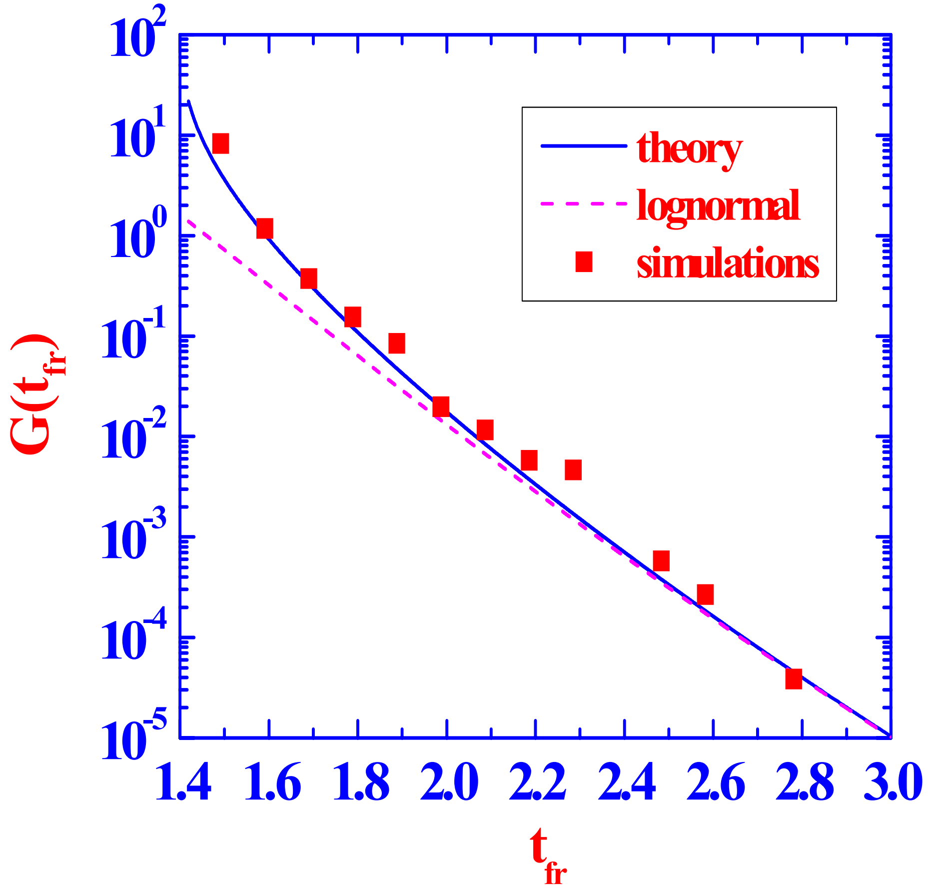
ε_q=0.5, ε=0.03, z=10, D=3, η(0)=1

ε_q=0.5, ε=0.05, z=10, D=3, η(0)=1

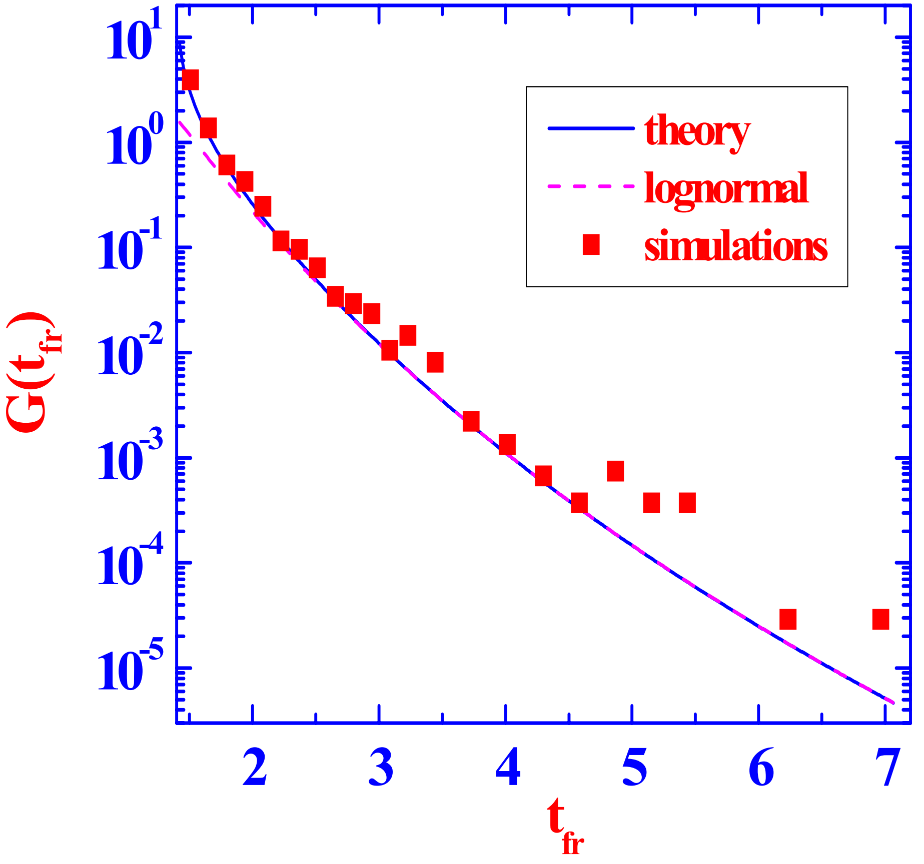
Numerical simulations

$G(t_{fr})$ - Distribution function of the front position

$\varepsilon_q=0.5$, $\varepsilon=0.03$, $z=10$, $D=3$, $\eta(0)=1$



$\varepsilon_q=0.5$, $\varepsilon=0.05$, $z=10$, $D=3$, $\eta(0)=1$



Other types of dissipative disorder

- Consider dynamics in the presence of a dissipative disorder $i\varepsilon\xi(z)|\psi|^{2n}\psi$

$$\text{near } \eta_m : \quad \frac{d}{dz}\ln\left[\ln\left(\frac{e^{2c_n}\delta\eta(z)}{2\eta_m}\right)\right] = 2^{n+1}\eta_m^{2n}\varepsilon\xi(z)$$

- Assuming $4D\varepsilon^2 z \ll 1$ and $\varepsilon \ll \varepsilon_q \ll 1$ we obtain

$$F(\delta\eta) \approx \frac{\exp\left\{-\ln^2\left[-\ln[(e^{2c_n}\delta\eta)/(2\eta_m)]/(2\tilde{c})\right]/(2^{2n+3}\eta_m^{4n}D\varepsilon^2 z)\right\}}{(32\pi D\varepsilon^2 z)^{1/2}\,2^n\eta_m^{2n}\delta\eta\,|\ln[(e^{2c_n}\delta\eta)/2\eta_m]|}$$

⇒ **F(η) has a diverging loglognormal form near η_m.**

- Similarly, $G(t_{fr})$ has a lognormal tail.

Relating the loglognormal divergence of F(η) to the asymptotic shape of the emerging front

- Considering disorder in the linear gain

$$\boxed{dt_{fr}/dz \approx \varepsilon\xi(z)t_{fr} \quad \Rightarrow \quad t_{fr}(z) \approx \mathrm{const} \times \exp[\varepsilon \mathrm{x}(\mathrm{z})]}$$

⇒ $G(t_{fr})$ is lognormal.

- For η near η_m and t>>1 the solitary waves have the following form

$$\boxed{\Psi_s(\mathrm{x}) \approx (\delta\eta/\eta_m)^{-1/4}\exp[-\eta_m t]}$$

Assuming t_{fr}>>1 and using the definition of t_{fr}

$$\boxed{t_{fr}(z) \approx -\ln(\delta\eta/\eta_\mathrm{m}) \quad \Rightarrow \quad \mathrm{x} \approx \ln[-\ln(\delta\eta/\eta_\mathrm{m})]/(2\varepsilon)}$$

⇒ The loglognormal divergence of F(η) is due to the form of the tail of the emerging front.

conclusions, part II

- Disorder in the linear/nonlinear gain coefficient leads to:

(a) loglognormal divergence of the amplitude distribution near the maximum possible amplitude, (b) lognormal tail for the front position distribution .

- The loglognormal divergence of the amplitude distribution is related to the asymptotic form of the emerging front tail.

- The disorder can lead to front formation even if the solitary wave is initially very far from the front state.

- Future work: consider other types of dynamics and other types of coherent structures.