Singularities, Shocks in Hele-Shaw flow

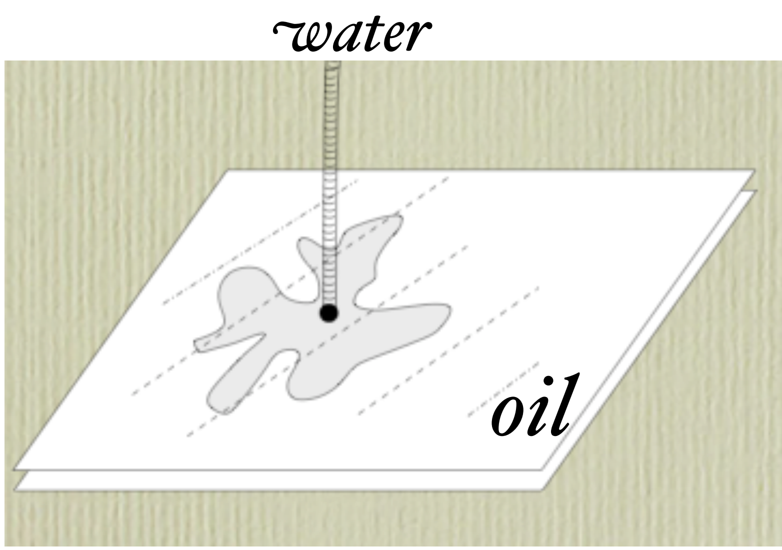
as

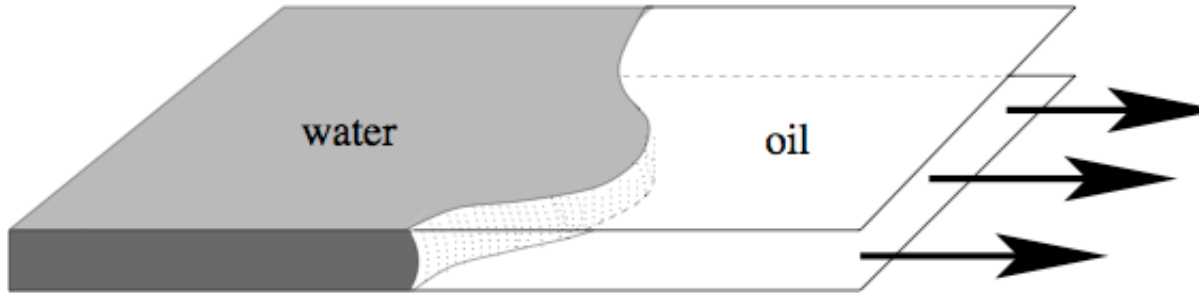
Stokes phenomena in Painlevé I

P. Wiegmann
University of Chicago

Vladimir Zakharov, 2010

Hele-Shaw cell (1894)

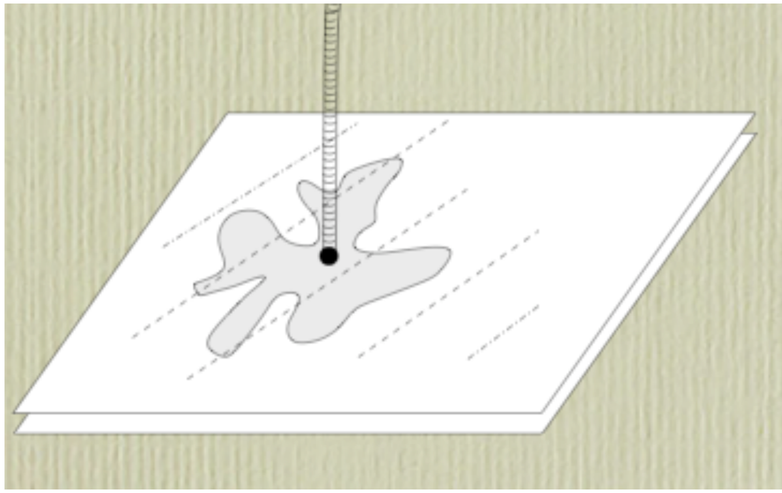




Oil *(exterior)*-incompressible liquid with high viscosity

Water *(interior)* - incompressible liquid with low viscosity

Vladimir Zakharov, 2010



$$\mathbf{v} = -\nabla p$$

Darcy Law

$$\Delta p = 0$$

velocity of a moving planar interface =
= a gradient of a harmonic field

$$p_{|\Gamma} = 0$$

$$p_{|\infty} = -\log|z|$$

Vladimir Zakharov, 2010

Painlevé I

$$u_{tt} - 6u^2 = \hbar t.$$

Physical Solution: $u \to +\sqrt{-\hbar t/6}$

$\Psi(X,t)$

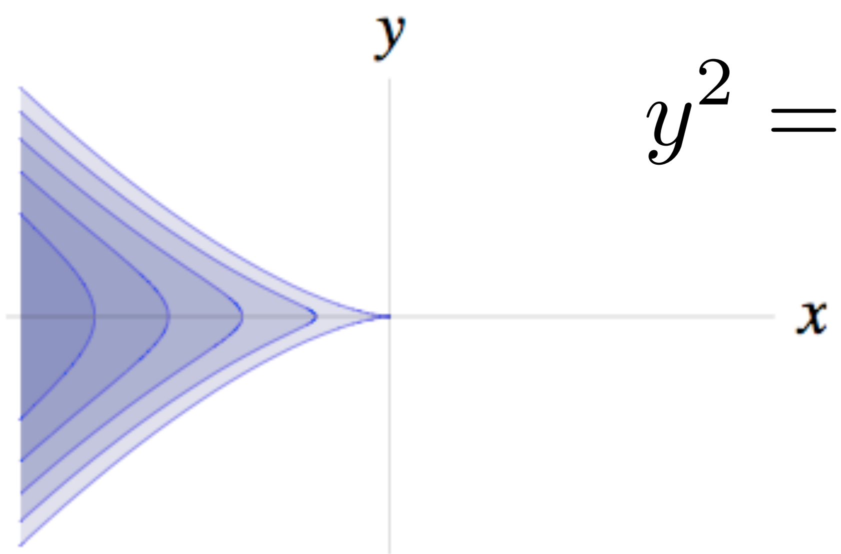
$$\Big(\partial_t^2 - 2u(t)\Big)\Psi = X\Psi$$

$$X = x + iy$$

$$T = \frac{\hbar t}{6}$$

Vladimir Zakharov, 2010

Finite time cusp singularities



$$y^2 = -(x + 2u(t))\Big(x - u(t)\Big)^2$$

$$u(t) = \sqrt{-t}$$

$$y^2 \sim x^3$$

Vladimir Zakharov, 2010

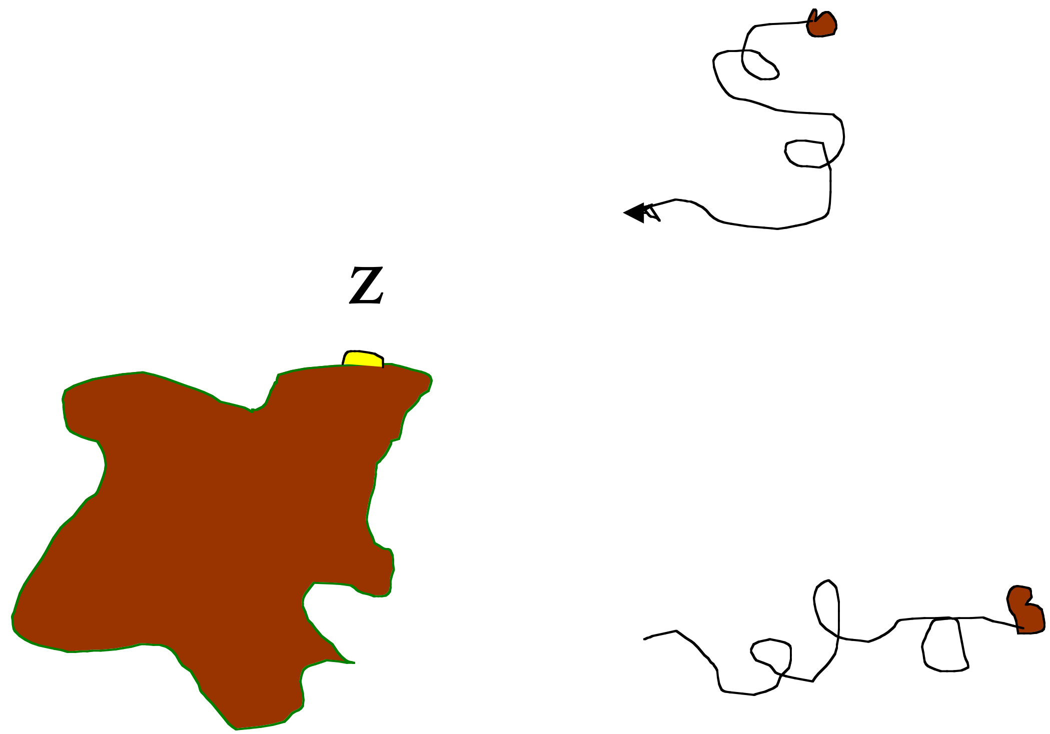
Brownian excursion with a moving boundary

A probability for BM to arrive on an element of the boundary is a *harmonic measure* of the boundary

$$\Delta p = 0, \quad p(z \to \infty) \to \log|z|$$

$$|\nabla p| - \text{ harmonic measure}$$

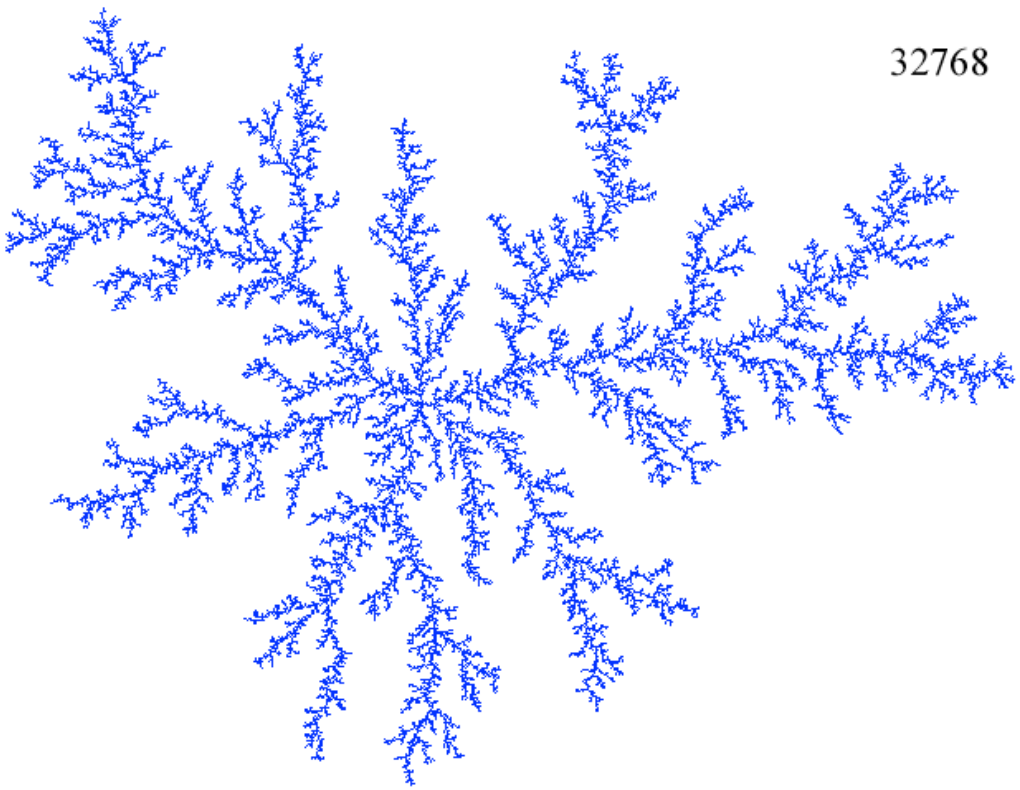
$$p = 0 - \text{ on the boundary}$$



Vladimir Zakharov, 2010

Continuous problem (a hydrodynamic limit):

a size of particles $\hbar \to 0$ tends to zero



zeros of $\Psi(X, t)$

The limit : *size of particles tends to zero is ill-defined*

Vladimir Zakharov, 2010

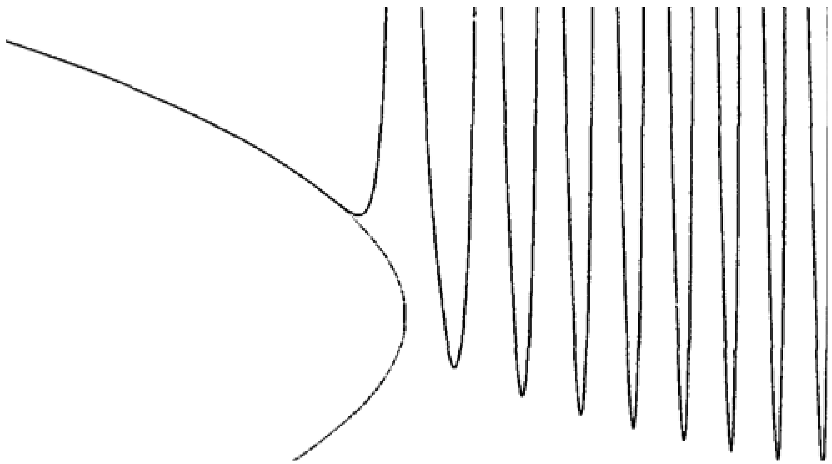
Weak solution

Darcy law is applied everywhere in the fluid except on

a moving, growing and branching graph of shocks,

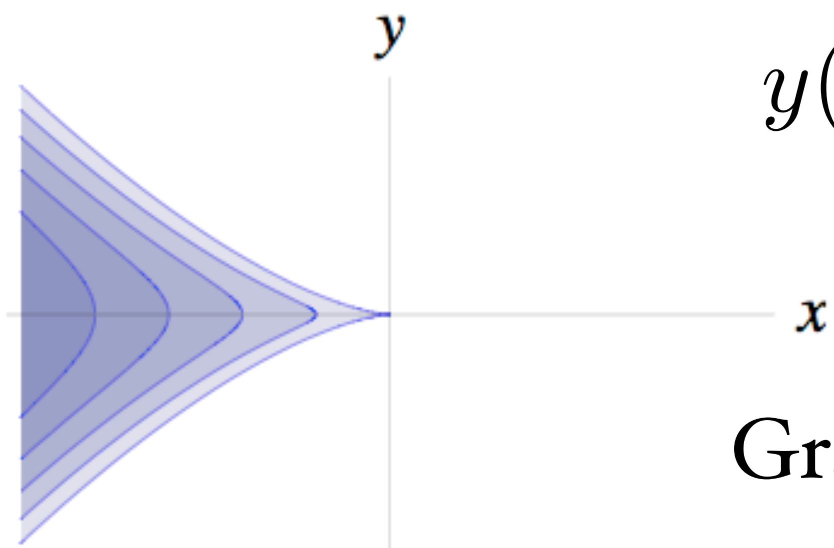
where pressure suffers a finite discontinuity.

$$u_{tt} - 6u^2 = \hbar t.$$



Vladimir Zakharov, 2010

Evolution of an elliptic curve



$$y(x) \to Y(X)$$

Graph to a Complex Curve

Darcy Law has a form of
Witham equation:

$$\partial_T Y = -\partial_X \phi,$$

$$\phi(X) = \psi + ip$$

Zabrodin, Krichever, P.W.

Vladimir Zakharov, 2010

Integral Form of the Darcy Law

$$\frac{d}{dT}\mathrm{Im}\oint_B(-iY)dX = \oint_B \mathbf{v}\times\mathbf{d}\ell = \oint_B d\psi = \text{flux of fluid},$$

$$\frac{d}{dT}\mathrm{Re}\oint_B(-iY)dX = -\oint_B \mathbf{v}\cdot\mathbf{d}\ell = -\oint_B dp = \text{vorticity}.$$

$$\mathrm{Re}\oint_{all\ cycles}(-iY)dX = 0$$

$$\mathrm{Im}\oint_\infty(-iY)dX = \text{time}$$

A unique weak solution

Vladimir Zakharov, 2010

$$\mathrm{Re} \oint_{all\ cycles} (-iY)dX = 0$$

$$\mathrm{Im} \oint_{\infty} (-iY)dX = \mathrm{time}$$

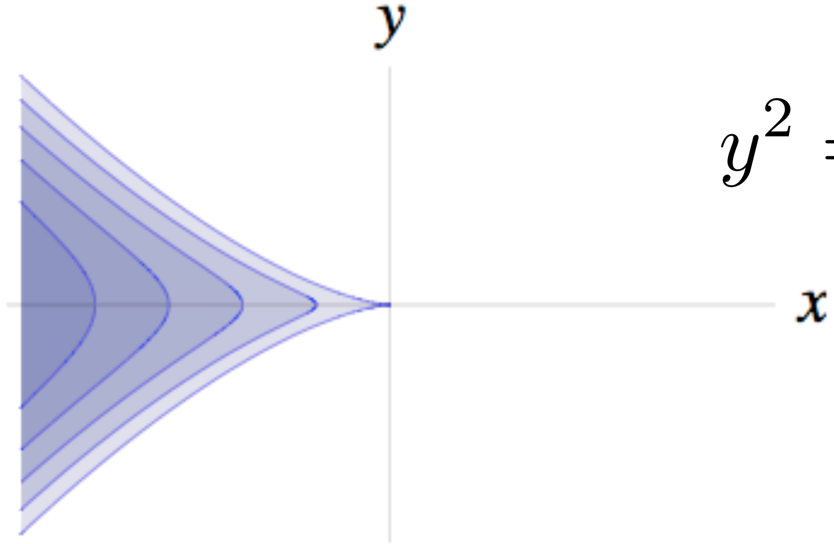
Boutroux condition/ Krichever

Vladimir Zakharov, 2010

Shocks and anti-Stokes lines

$$\mathrm{Re}\int^{X}(-iY)dX = 0, \quad X \in \text{shocks}$$

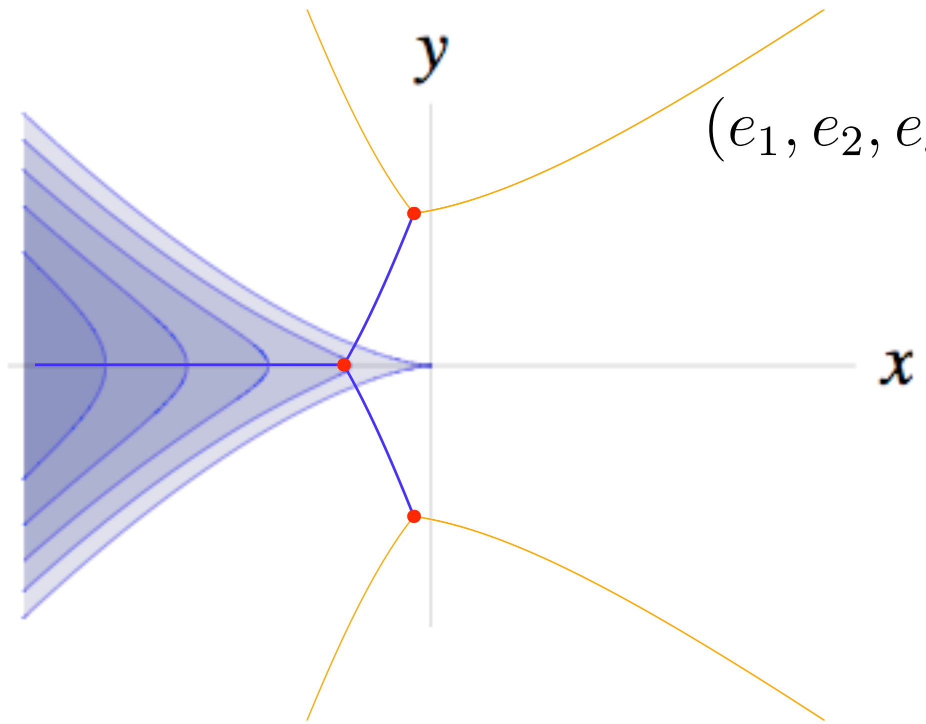
$$\mathrm{Re}\int^{X}(-iY)dX > 0, \quad X \to \text{shocks}$$



$$y^2 = -(x+2u(t))\Big(x-u(t)\Big)^2$$

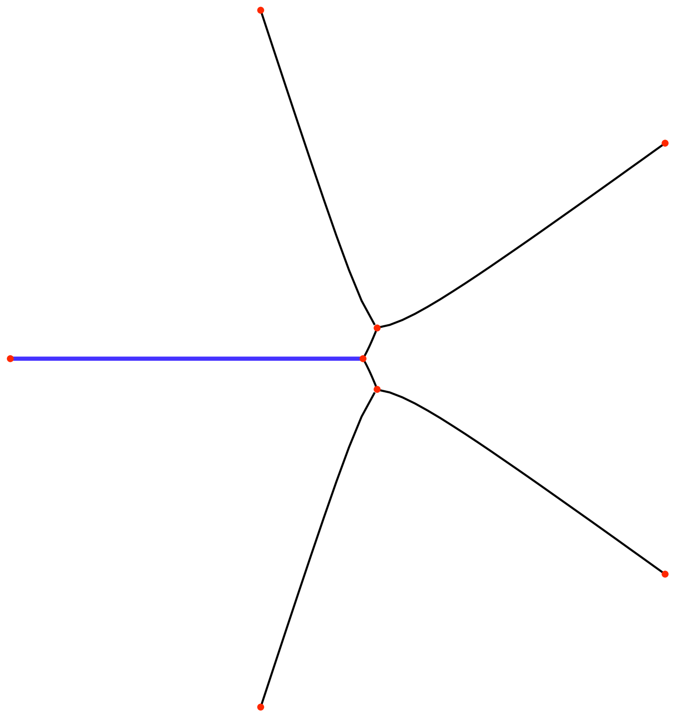
$$u(t) = \sqrt{-t}$$

$$y^2 = (x-e_1)(x-e_2)(x-e_3)$$

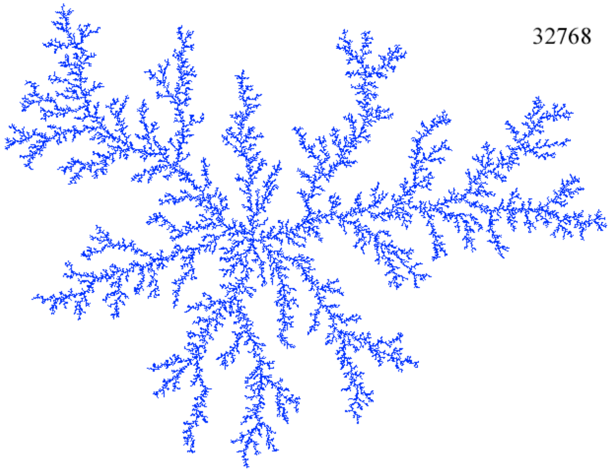


$$(e_1, e_2, e_3) = \sqrt{\frac{3}{h^2 - \frac{3}{4}}}(-1, \tfrac{1}{2} + \mathrm{i}h, \tfrac{1}{2} - \mathrm{i}h)\sqrt{t}$$

$$h \approx 3.24638225374427887 5676.$$



Two generations of branching



32768 generation of branching

Stokes Phenomenon in Painlevé I

Painlevé I $\qquad u_{tt} - 6u^2 = \hbar t.$

$$\left(\partial_t^2 - 2u(t)\right)\Psi = X\Psi$$

$$Y(X,t) = \lim_{\hbar\to 0}(i\hbar\partial_X)\log\Psi(X,t)$$

$$\hbar\log\Psi|_{X\to\infty} \to \frac{4}{5}X^{5/2} - t\,X^{1/2}, \quad |\mathrm{Arg}|X < \pi.$$

$$\psi_{\pm}^{WKB}(X,t) \sim e^{\pm \frac{1}{\hbar}\int^{X}(-iY)dX}$$

Time-independent monodromy data:

$$\mathrm{Re}\oint_{all\ cycles}(-iY)dX = 0$$

$$\mathrm{Im}\oint_{\infty}(-iY)dX = \mathrm{time}$$

Boutroux, Krichever, Novikov-Grinevich

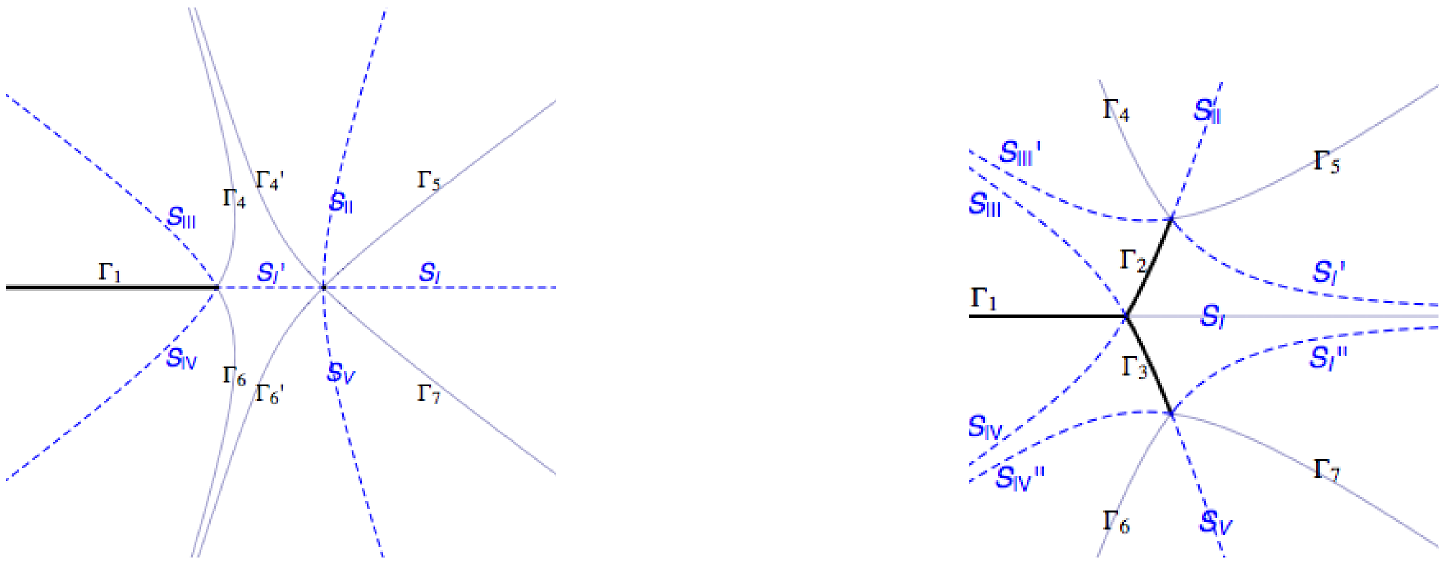
identical to the Darcy law

Darcy law requires the asymptote

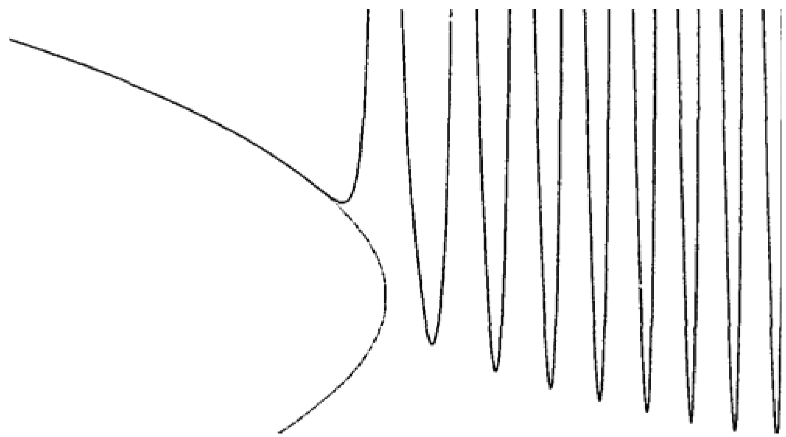
$$\hbar \log \Psi|_{X\to\infty} \to \frac{4}{5} X^{5/2} - t\, X^{1/2}, \quad |\mathrm{Arg}| X < \pi.$$

$$\Psi^{WKB} = A_S\, \psi_+^{WKB} + B_S\, \psi_-^{WKB},$$

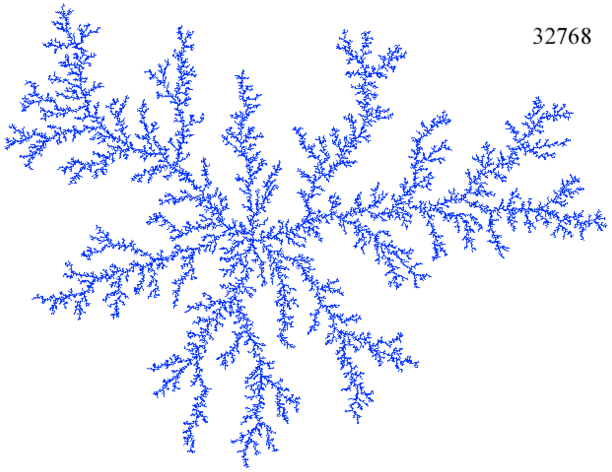
$$\psi_\pm^{WKB}(X,t) \sim e^{\pm \frac{1}{\hbar} \int^X (-iY) dX}$$



Position of zeros of the physical wave-function



Smooth and oscillatory regimes of the Painleve function



<u>Hypothesis</u>:

The pattern is Stokes lines of higher order Painleve equations.