Inertial particles in a telegraph flow

G. Falkovich, S. Musacchio, L. Piterbarg and MV

Weizmann Institute of Science

University of Southern California

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Transport in fluids

Infinitesimal fluid tracers move with the fluid's velocity

$$\dot{\boldsymbol{R}}(t) = \dot{\boldsymbol{X}}_1(t) - \dot{\boldsymbol{X}}_2(t) = \Delta \boldsymbol{U}(\boldsymbol{X}, t)$$

Real particles have finite size and density mismatch with the fluid - **inertial particles**

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{V}(t)$$

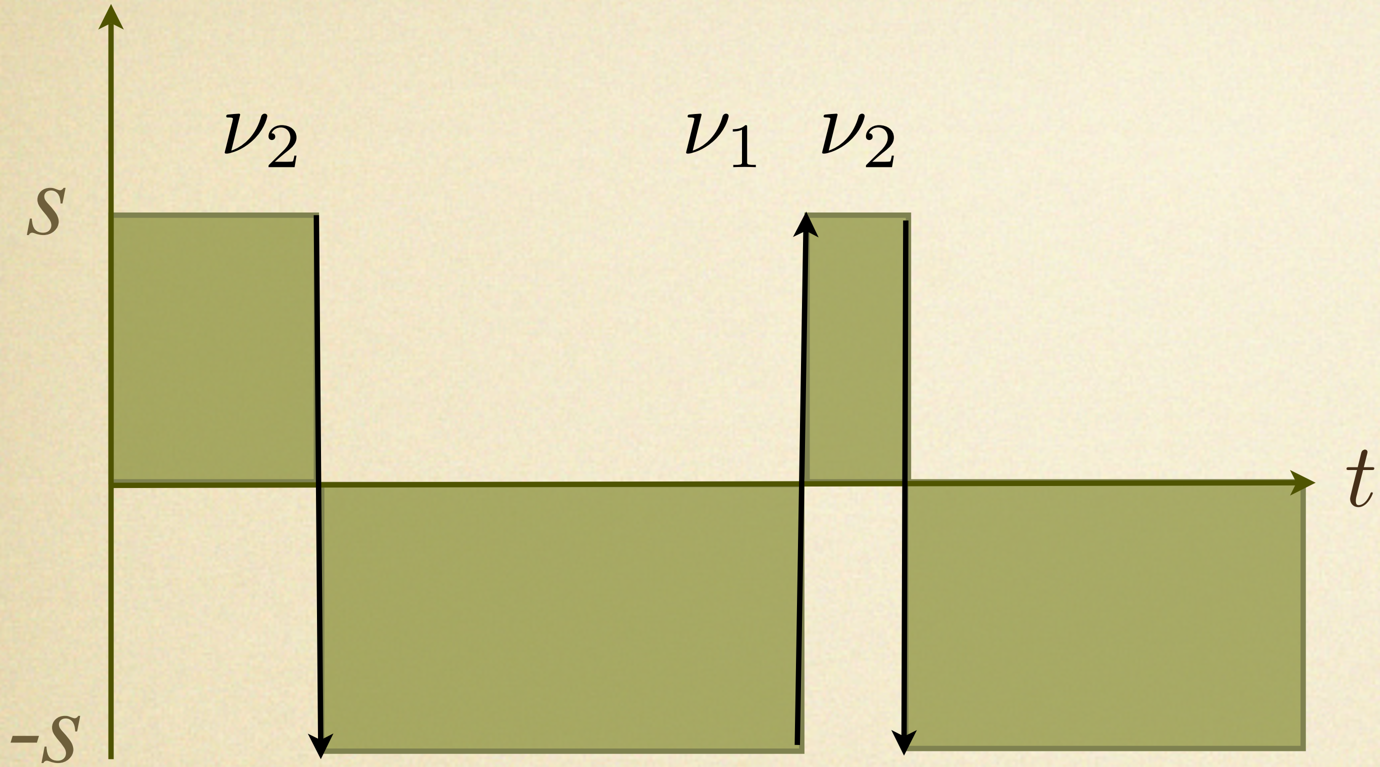
$$\dot{\boldsymbol{V}}(t) = -\frac{1}{\tau}\left(\boldsymbol{V}(t) - \Delta \boldsymbol{U}(\boldsymbol{X}, t)\right) \qquad \tau \equiv \frac{2}{9}\frac{\rho_p a^2}{\eta}$$

$$\mathrm{St} \equiv |\partial \Delta U|_{\mathrm{characteristic}}\tau = s\tau$$ Stokes number

Time-correlations of the flow and clustering

- Spontaneous formation of clusters of particles suspended in chaotic flows may originate from flow compressibility or particle inertia.
  - Compressibility - particles are trapped in regions of ongoing compression
  - Inertia - inertia causes their ejection from vortical regions.
- Weak inertia in incompressible flow - similar tracer dynamics in weakly compressible flows. Both gave been studied for delta-correlated flows.
- **How do time-correlations of the flow change the qualitative picture?**

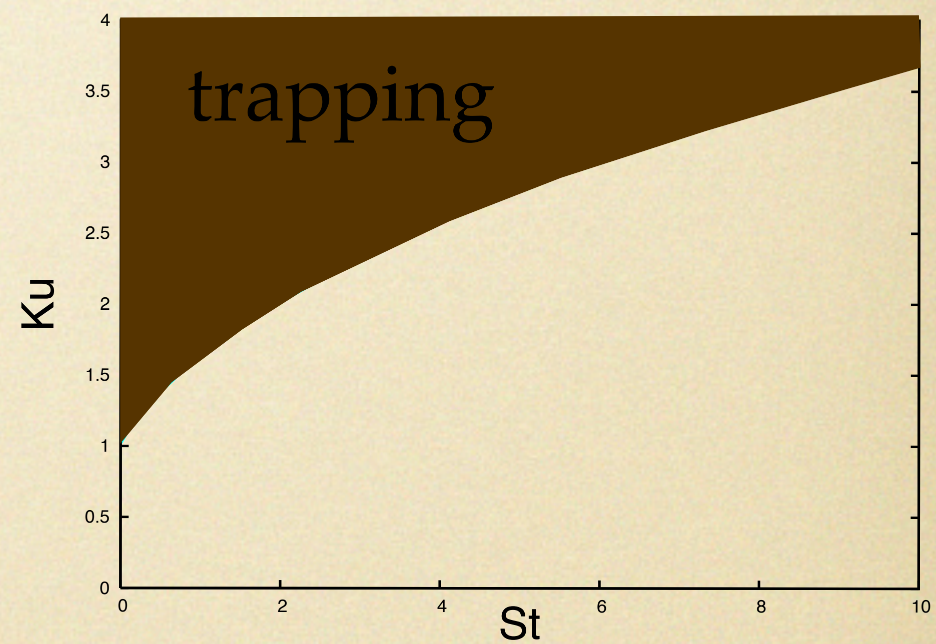
Telegraph Noise



$$\langle s^2(t)\rangle = s^2$$

$$\tilde{s}(t) \equiv s(t) - \langle s(t)\rangle$$

$$\langle \tilde{s}(0)\tilde{s}(t)\rangle \propto e^{-(\nu_1+\nu_2)t}$$



Stokes number $\quad \mathrm{St} \equiv s\tau$

Kubo number $\quad \mathrm{Ku} \equiv \dfrac{s}{\nu_1+\nu_2}$

- statistical homogeneity and isotropy of the flow $\quad \langle R\rangle = \mathrm{const}$
- particles spend more time in regions of local compression $\quad \nu_2 - \nu_1 > 0$

Formulae of differentiation

V. E. Shapiro and V. M. Loginov (1978)

$$\dot{\sigma}(t) = -\sigma^2 - \frac{1}{\tau}\sigma + \frac{s}{\tau}, \quad \sigma(t) \equiv \frac{V(t)}{R(t)}$$

$$x \equiv \sigma + \frac{1}{2\tau} \qquad \dot{x} = A(x,t) + B(x,t)\tilde{s}(t)$$

$W(x,t)$ probability of finding x at time t

$W_t + [AW + B\tilde{s}W]_x = 0$ Liouville equation

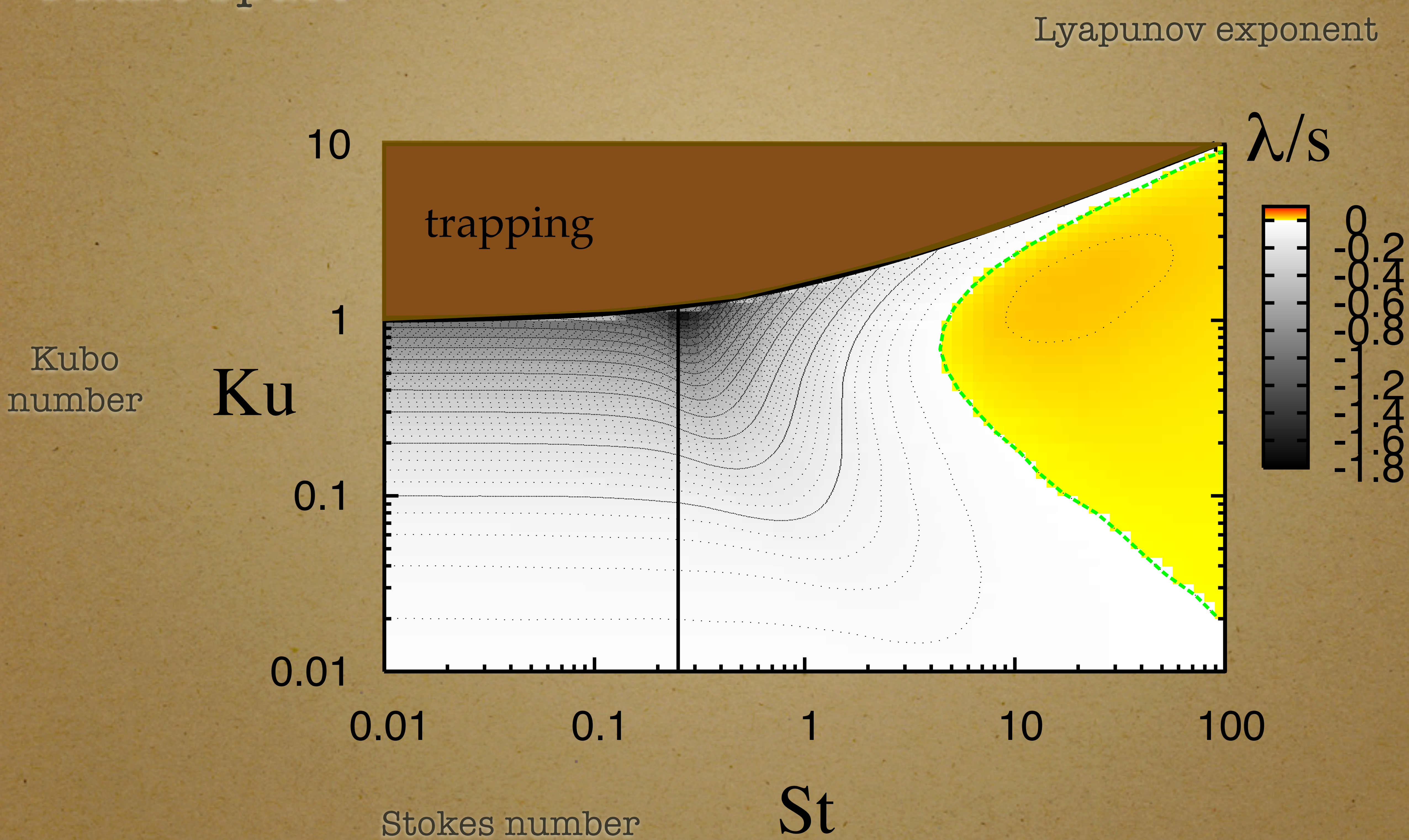
$$\boxed{\langle \tilde{s}W \rangle_t - \langle \tilde{s}W_t \rangle + \nu \langle \tilde{s}W \rangle = 0}$$

$$\tilde{s} \equiv s - \langle s \rangle$$

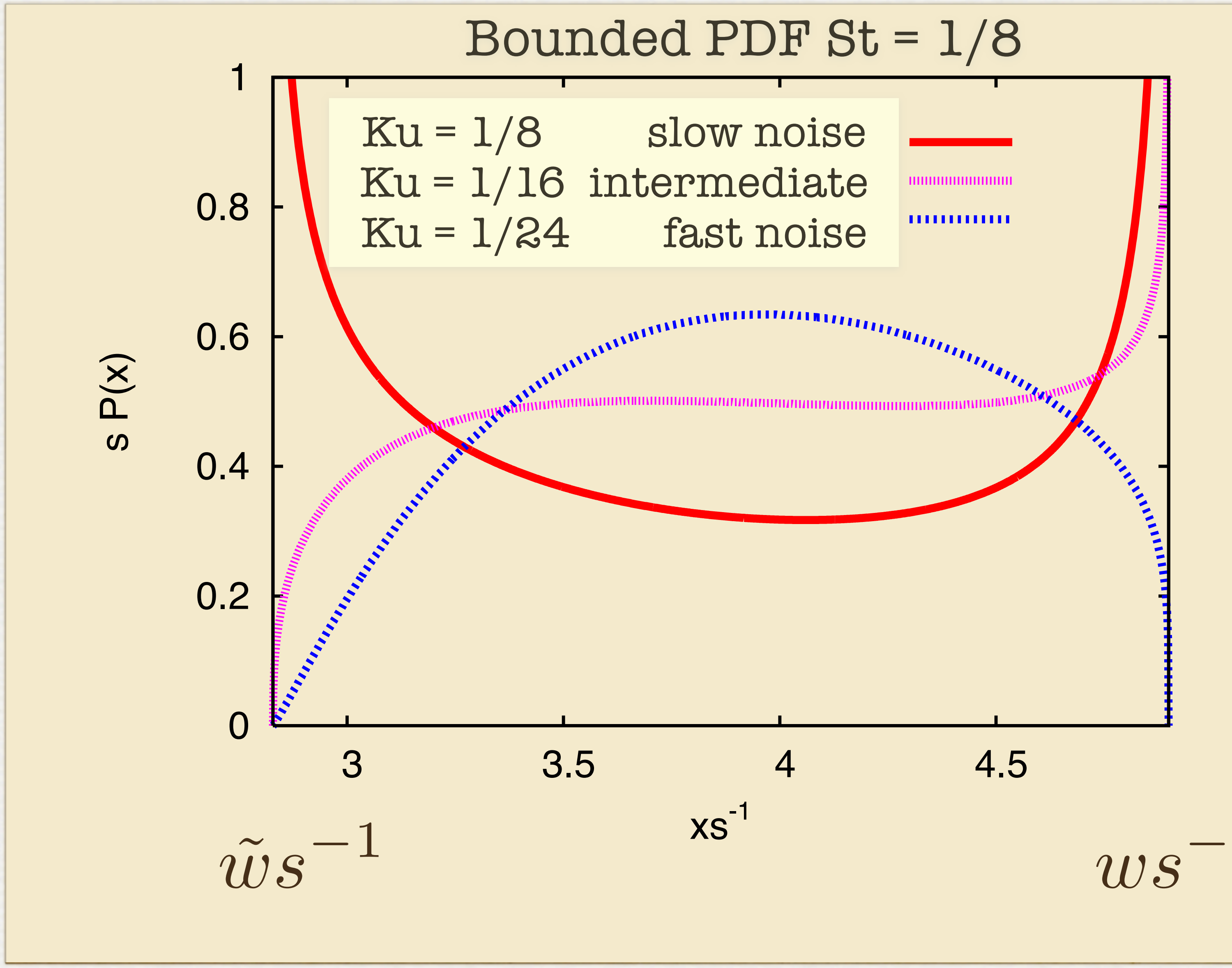
$$\langle W \rangle_t + \left[\frac{1}{4\tau^2} - x^2 - \frac{\langle s \rangle}{\tau} \langle W \rangle \right]_x + \frac{1}{\tau} \langle \tilde{s}W \rangle_x = 0$$

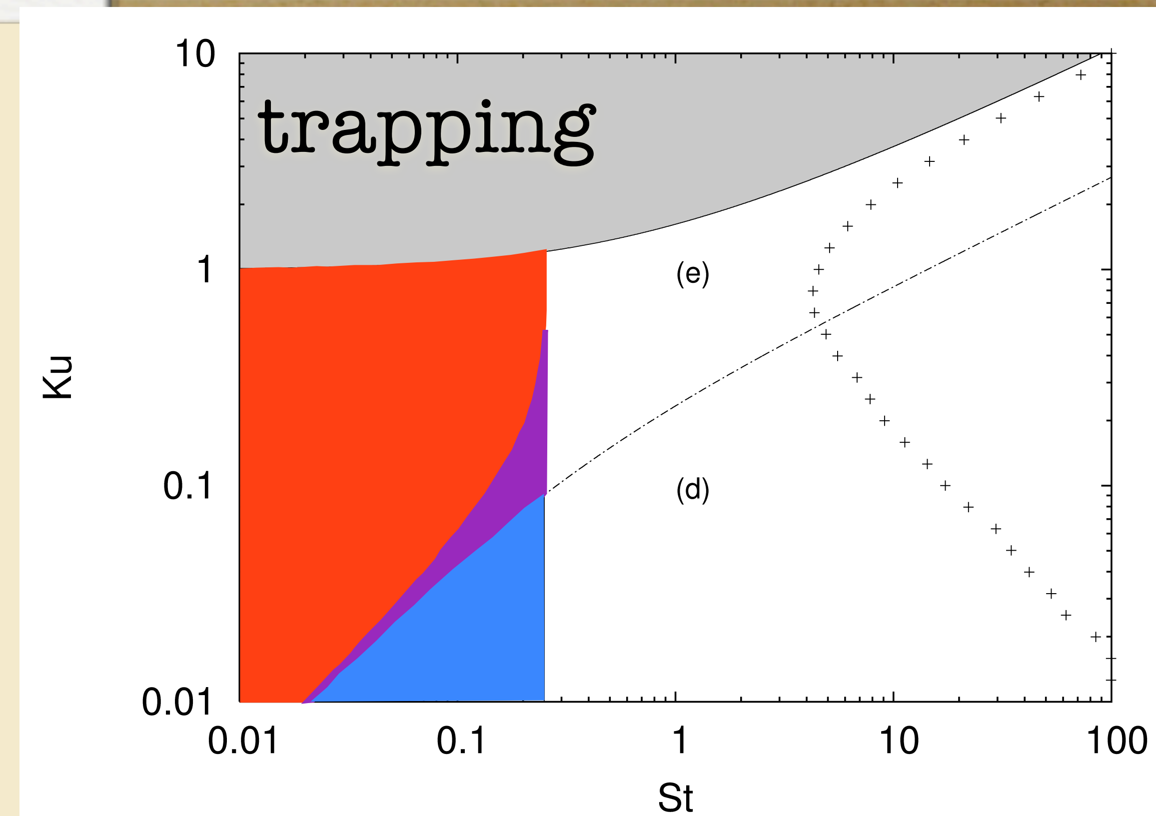
$$\langle \tilde{s}W \rangle_t + \left[\left(\frac{1}{4\tau^2} - x^2 - \frac{\langle s \rangle}{\tau} \right) \langle \tilde{s}W \rangle \right]_x + \frac{s^2 - \langle s \rangle^2}{\tau} \langle W \rangle_x + (\nu_1 + \nu_2) \langle \tilde{s}W \rangle = 0$$

Phase space



Chaotic regime observed in the range of Kubo numbers where time correlations are **long enough for the so that particles feel significant stretching** and **short enough so that they do not get trapped.**





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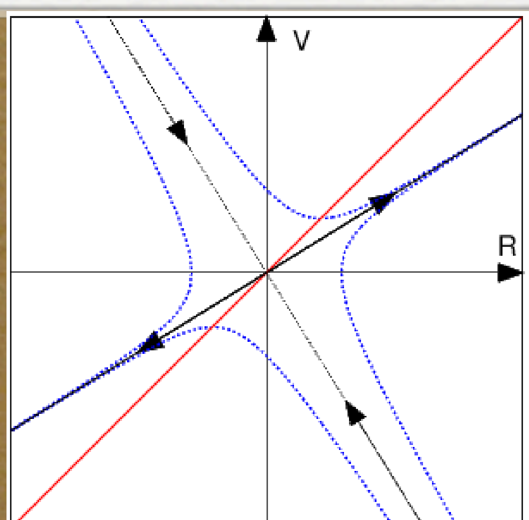

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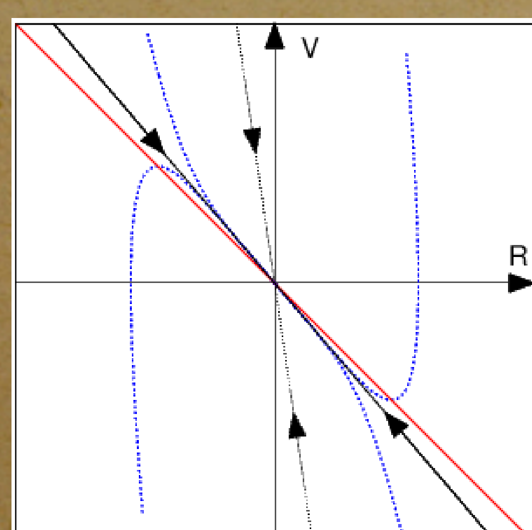
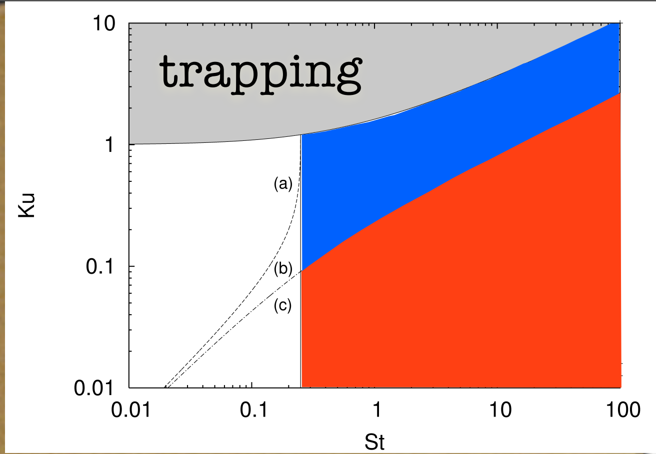


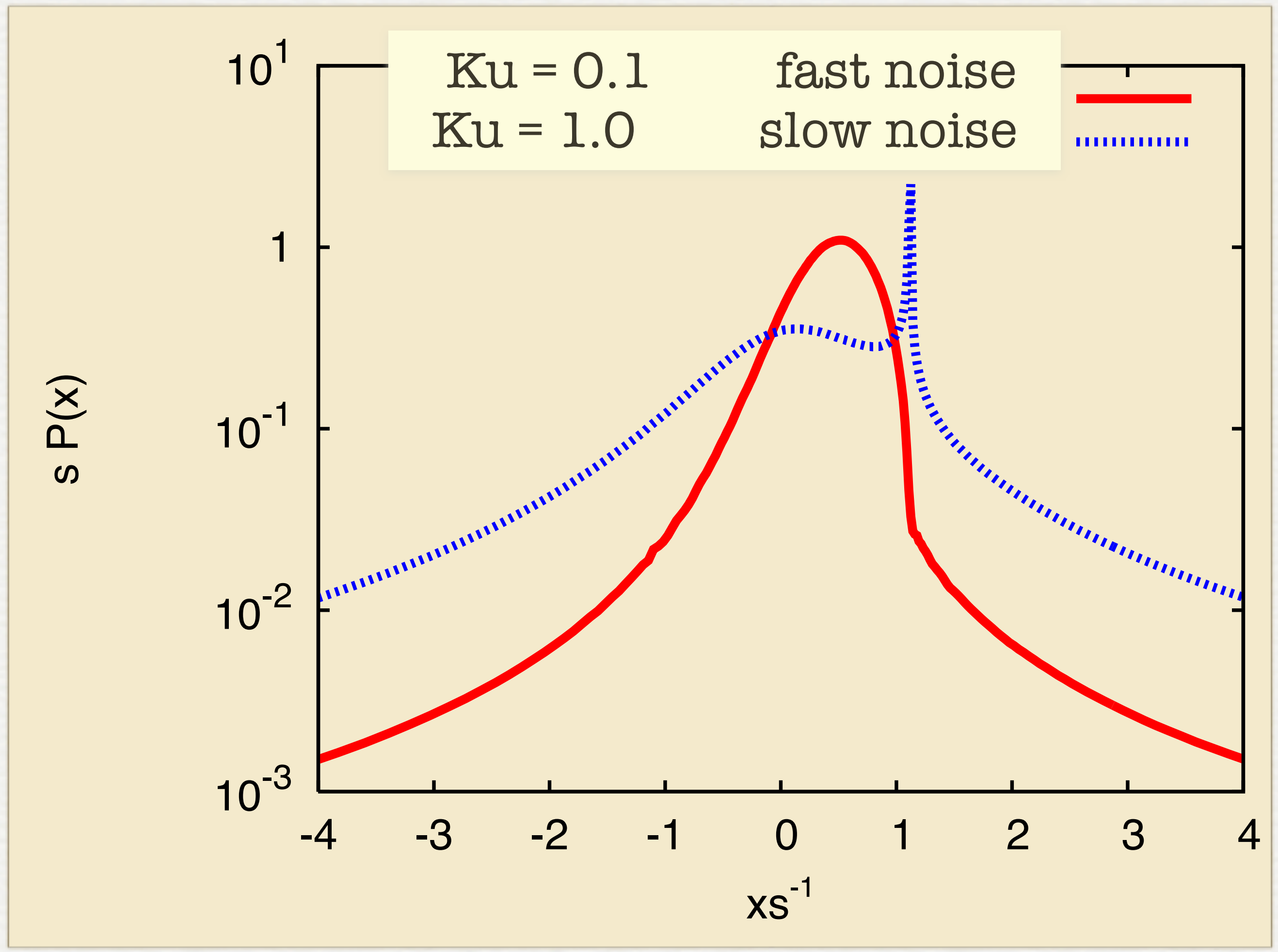
$$p = C\,\frac{(w-x)^{m-1}(x-\tilde{w})^{\tilde{m}-1}}{(w+x)^{m+1}(x+\tilde{w})^{\tilde{m}+1}}$$

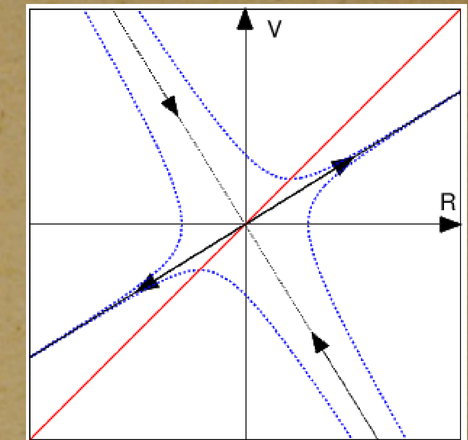
$$w = \sqrt{1+4\mathrm{St}}/2\tau\,, \qquad m = \nu_2/2w$$

$$\tilde{w} = \sqrt{1-4\mathrm{St}}/2\tau\,, \qquad \tilde{m} = \nu_1/2\tilde{w}$$



Unbounded PDF for St > 1/4





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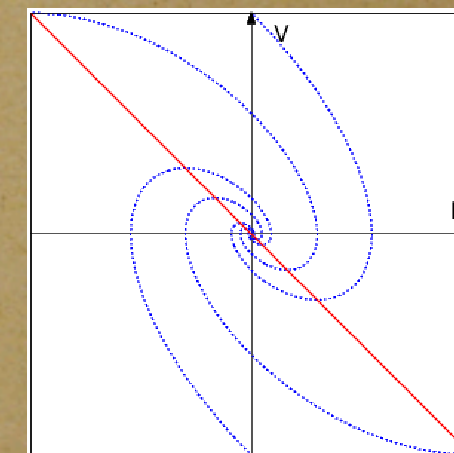

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tails:

$$p \sim \frac{1}{x^2}$$

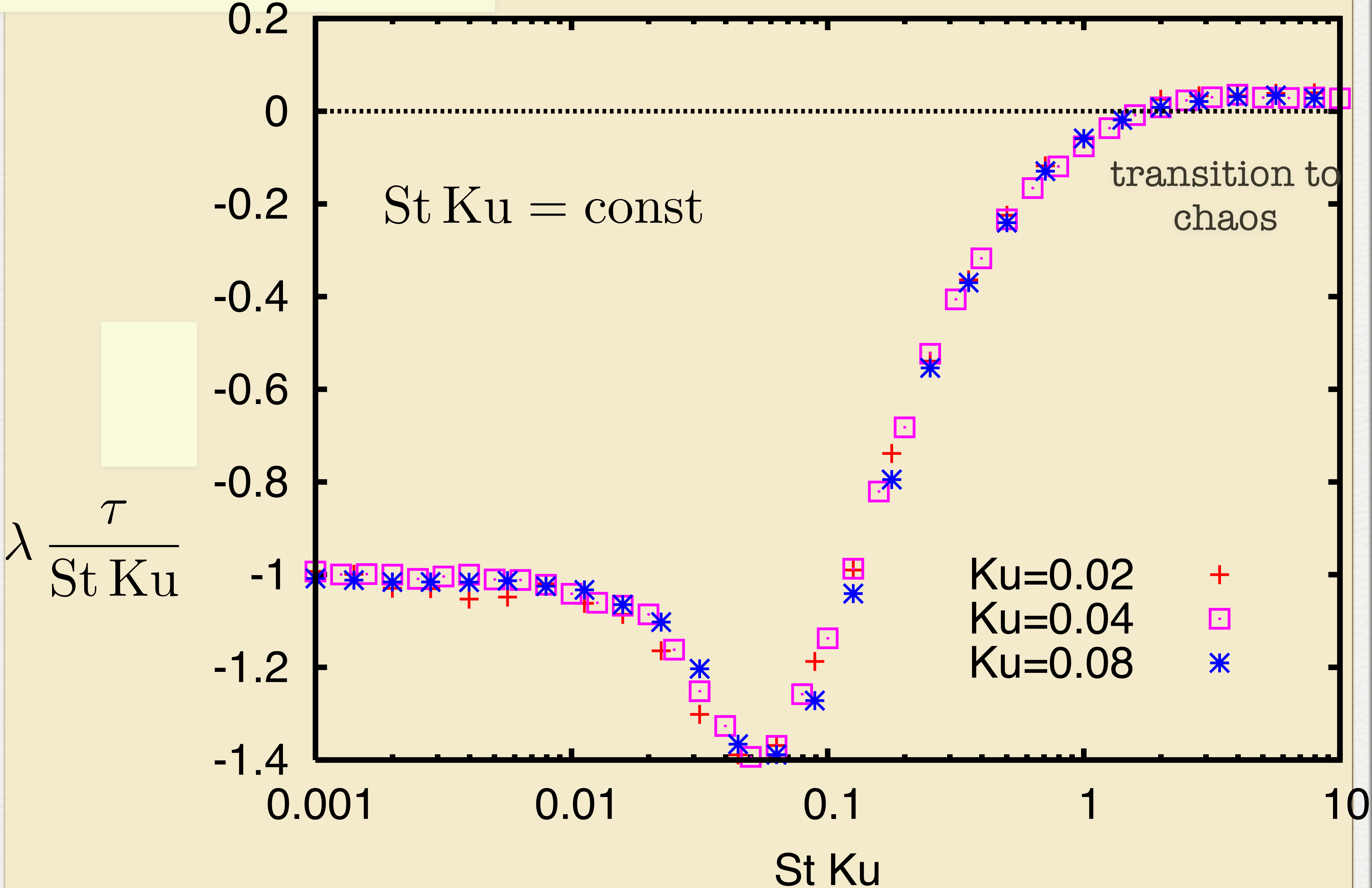
$$p = \frac{|w-x|^{m-1}}{|w+x|^{m+1}(x^2+\tilde{w}^2)} \exp\left[\tilde{m} \arctan\left(\frac{x}{\tilde{w}}\right)\right] (C_1 I_1 + C_2 I_{w,\infty}(x))$$

$$I_1 \equiv \int_{-w}^{x} dy \frac{|w+y|^{m+1}(\nu_1+\nu_2-2y)}{|y-w|^{m-1}(w^2-y^2)} \exp\left[\tilde{m} \arctan\left(\frac{y}{\tilde{w}}\right)\right]$$

$$w = \sqrt{4\mathrm{St}+1}/2\tau\,, \quad m = \nu_2/2w$$

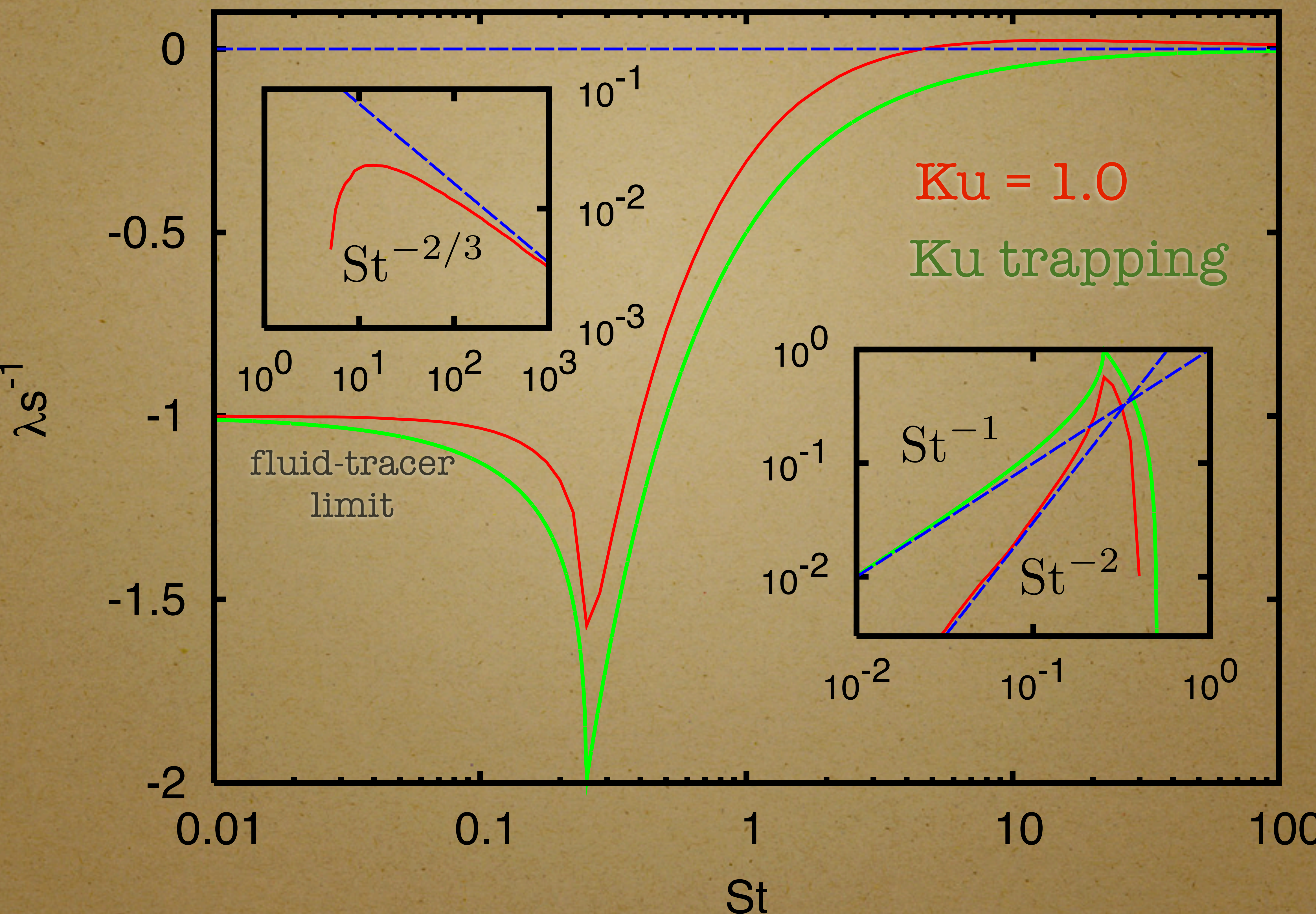
$$\tilde{w} = \sqrt{4\mathrm{St}-1}/2\tau\,, \quad \tilde{m} = \nu_1/2\tilde{w}$$

Short-correlated limit



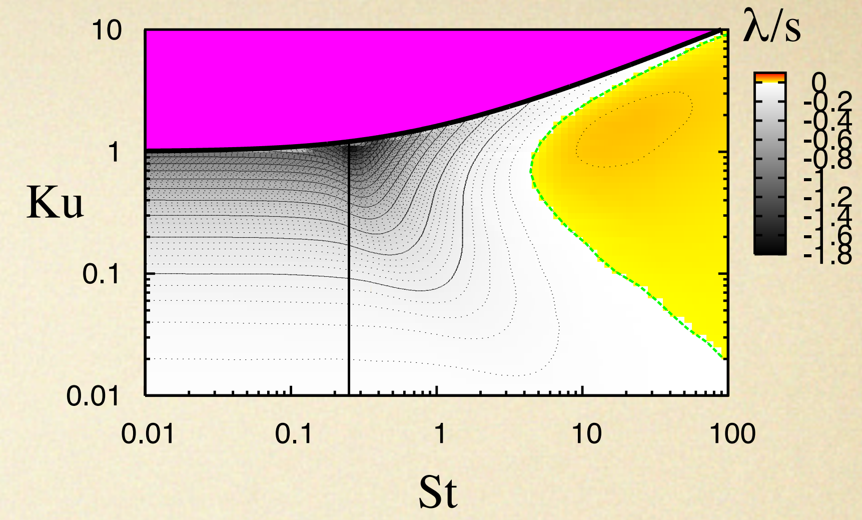
$$\lim_{\nu_1+\nu_2\to\infty} \langle s(t)\rangle = \begin{cases} -D & \text{fluid tracers} \\ 0 & \text{inertial particles} \end{cases}$$

Long-correlated flows



$St^{-2/3}$ has been shown for delta-correlated flows

Telegraph model



- A simple model that introduces time correlations, reproduces the phenomenon of trapping particles in compressing regions for long-correlated flows.

- For large St we found a chaotic region in St-Ku plane. Thus inertia is responsible for transition from strong clustering to chaotic regime.

- In order to describe preferential concentration one needs to go to 2D.

- Note that Lyapunov statistics is able to describe the clustering of inertial particles at dissipative scales of the turbulent flow, but not within the inertial range.