# Chain rule for scattering matrices and modified analytic perturbation procedure

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# Abstract

H. Poincare, in his book on celestial mechanics, [13], conjectured that convergence of the analytic perturbation procedure for a classical system containing small denominators can be improved via “elimination of dangerous resonances” (that is small denominators). This idea of Poincare inspired physicists to develop a modified analytic perturbation procedure in quantum mechanics based on special selection of the first order approximation in the form of a solvable model and completed by the standard analytic approximation procedure connecting the model to the perturbed Hamiltonian,see [15], where the modified first order approximation is selected as a function of the unperturbed Hamiltonian. I. Prigogine failed attempting to use the same approach in resonance problems. We suggest a resonance modification of the Poincare idea selecting the first order approximation as a fitted zero-range model of the perturbed Hamiltonian.

# Modified two-steps analytic perturbation procedures

The standard analytic perturbation procedure, see [17],is convergent for the eigenvalue of the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon V$:

$$\lambda_1(\varepsilon) = \lambda_1(0) + \varepsilon\mu_1^1 + \varepsilon^2\mu_2^1 + \varepsilon^3\mu_3^1 + \dots$$

if the magnitude of the perturbation $\varepsilon V$ is dominated as $2|\varepsilon V| < \rho_1$ by the spacing $\rho_1 = \min_{s\neq 1} |\lambda_1(0) - \lambda_s(0)|$ of the corresponding eigenvalue $\lambda_1(0)$ of $H_0$. For a dense discrete spectrum this condition guarantees convergence only for particularly small values of $\varepsilon$, and does not provide any spectral conditions of convergence in resonance problems, with embedded eigenvalues see, for instance [5, 6]. Demand on analysis of the resonance problems was growing during the previous century, beginning from the acoustic scattering by the Helmholtz Resonator, [10], and ending by the problem of scattering electrons with Fano resonances [7, 8, 9] or scattering by metal-molecular contacts, see [19, 20].

# Modified two-steps analytic perturbation procedures.

The elegant idea of the two-steps analytic perturbation procedure was practically implemented by T. Murota and J. Dollard, see [14, 15], who fought with it the infrared divergence in scattering by Coulomb potential. The corresponding perturbed Hamiltonian in the Interaction representation

$$-\frac{1}{2m}\Delta+\frac{\alpha}{|x|}\xrightarrow{I}-\frac{1}{2m}\Delta+\frac{\alpha}{|p|t|/m+x|}$$

can be represented as a sum of the re-normalized unperturbed Hamiltonian $\tilde{H}_0^{\alpha}$ corrected by a relatively small perturbation $\tilde{V}$:

$$-\frac{1}{2m}\Delta+\frac{\alpha}{|p|t|/m+x|}=\left[-\frac{1}{2m}\Delta+\frac{\alpha m}{|t||p|}\right]+\underbrace{\left[\frac{\alpha}{|p|t|/m+x|}-\frac{\alpha m}{|t||p|}\right]}_{(1)}\equiv$$

# Modified two-steps analytic perturbation procedures.

To overcome the problem of infrared divergences they redefined the free Hamiltonian $\hat{H}_0$ [14, 15] in the interaction representation $I$ as

$$\hat{H} = \hat{H}_0 + \hat{V} = -\frac{1}{2m}\Delta + \frac{\alpha}{|x|} \xrightarrow{I} -\frac{1}{2m}\Delta + \frac{\alpha}{|p|t|/m + x|} =$$

$$\left[-\frac{1}{2m}\Delta + \frac{\alpha m}{|t||p|}\right] + \left[\frac{\alpha}{|p|t|/m + x|} - \frac{\alpha m}{|t||p|}\right] \equiv \tilde{H}_{\tilde{I}} + \tilde{V}_{\tilde{I}}. \qquad (2)$$

# Modified two-steps analytic perturbation procedures.

Here

$$\tilde{H}_{\tilde{I}} \equiv \left[ -\frac{1}{2m}\Delta + \frac{\alpha m}{|t||p|} \right],$$

$$\hat{V}_{\tilde{I}}(\hat{\mathbf{r}}(t)) = \frac{\alpha}{|\hat{\mathbf{p}}t/m + \hat{\mathbf{r}}|} - \frac{\alpha m}{|t||\hat{\mathbf{p}}|} \approx \text{Const}\frac{\ln|t|}{|t|^2} \tag{3}$$

The evolution is given, in the interaction representation, in terms of $\tilde{V}_{\tilde{I}}$:

$$\tilde{U}_{t,t'} = \exp[i\tilde{H}_0 t]\exp[-i\hat{H}(t-t')]\exp[-i\tilde{H}_0 t'], \quad \tilde{S} = \tilde{U}_{\infty,-\infty}. \tag{4}$$

# Modified two-steps analytic perturbation procedures.

Existence of strong limits for the evolution operators $\tilde{U}_{0,t'}$, $t' \to -\infty$, $\tilde{U}_{t,0}$, $t \to \infty$t was shown in [15] which implies existence of $\tilde{S}$- matrix

$$\tilde{S} = T \exp[-i \int_{-\infty}^{\infty} dt \tilde{V}_{\tilde{I}}(\mathbf{r}(t))]. \tag{5}$$

Here $\tilde{I}$ means new interaction representation with $\hat{V} \to \tilde{V}$ substituted by $\tilde{V}_{\tilde{I}}$.

# Modified two-steps analytic perturbation procedures.

The Murota-Dollard idea of the two step modification of the analytic perturbation procedure in spectral analysis and scattering theory attracted attention of physicists, see for instance the paper [16], where a general procedure of elimination of collinear divergences is suggested for a representative class of gauge-invariant Hamiltonians in QED. I. Prigogine, see [11], probably also inspired by Murota-Dollard success, attempted to implement the H. Poincare idea of elimination of dangerous resonances in *quantum-mechanical resonance scattering problem* via construction an *intermediate operator* $H_0^\varepsilon$ - an analog of $\tilde{H}_{\tilde{I}}$ for a two-step analytic perturbation procedure $H_0 \to H_0^\varepsilon \to H_\varepsilon$ - as a function of the unperturbed one: $H_0^\varepsilon = \Phi_\varepsilon(H_0)$. Since the twenty-years-search of such an operator yielded no results, I. Prigogine declared finally that an intermediate operator with the expected properties does not exist.

# Jump-start modification of the analytic perturbation procedure

In the papers [1, 3, 12] the modified modified analytic perturbation procedure is suggested for a junction in the quantum network. The central point of the procedure described in the papers was the search for the solvable generator responsible for slow scattering processes. To avoid confusion with I. Prigogine's construction described above, our version of the corresponding Intermediate operator was called *Jump-start* see [4]. Contrary to the Murota-Dollard situation, the scattering matrix of the perturbed Hamiltonian exists, but the question of effective calculation of it is not trivial, because it involves perturbation of the eigenvalues of the unperturbed Hamiltonian embedded into the continuous spectrum. The perturbation transforms the embedded eigenvalues into complex resonances, which may not even admit a spectral interpretation.

# Jump-start modification of the analytic perturbation procedure

Consider a junction with several leads of equal width $\delta$, attached orthogonally to the flat portions of the piecewise smooth boundary of a quantum well. The Hamiltonian of the junction is a one-electron Schrödinger operator on the junction with homogeneous Dirichlet condition on the boundary. The corresponding scattering matrix exists, but it can't be calculated based on spectral data of the quantum well $\Omega_{in}$ via the standard analytic perturbation techniques. In [12]. If $\mathcal{DN}$ is the relative Dirichlet-to-Neumann map $\mathcal{DN}_\Gamma$ calculated based on the solution $u$ of the the boundary problem

$$-\Delta u = \lambda u = 0,\ u\Big|_{Gamma} = u_\Gamma \in W_2^{3/2}(\Gamma),\ u\Big|_{\partial\Omega\setminus\Gamma} = 0,$$

with an additional Meixner condition $u \in W_2^2(\Omega_{in}) \cap W_2^1(\Omega_{in})$:

$$\mathcal{DN}_\Gamma\ u_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma \in W_2^{1/2}(\Gamma).$$

# Jump-start modification of the analytic perturbation procedure

Hereafter we impose the following assumptions, used in [1, 12], on the geometrical and physical characteristics of the junction. We assume that:

1. The 2d leads $\omega_s$ are equivalent, width $\delta$ and semi-infinite. The potential of the Schrödinger operator is constant $V\omega = V_\omega$ on the links and real and piecewise continuous on the whole junction.

2. The scaled Fermi level $\Lambda_F = 2mE_F\hbar^2$ of the junction lies on the first spectral band in the leads: $\pi^2\delta^{-2} < \Lambda_F < 4\pi^2\delta^{-2}$. Hence the first spectral band is the conductivity band of the junction.

3. There is only one simple eigenvalue $\lambda_1$ of the Dirichlet Schrödinger operator on $\Omega_{in}$, with Meixner condition imposed on the domain of the operator, $\int_{\Omega_{in}} |\nabla u|^2 dx < \infty$. The corresponding eigenfunction is denoted by $\varphi_1$.

# Jump-start modification of the analytic perturbation procedure

Note that the Dirichlet Schrödinger operator on the leads, with Rashba spin-orbital interaction, admits separation of longitudinal and cross-section variables and the spectral analysis in the leads can be developed based on the separation for Rashba interaction. We postpone to an incoming more technical paper an extended discussion of the electron transport in the junction with regard to the spin. The spectral parameter of the Schrödinger operator is hereafter scaled as $\lambda = 2mE\delta^{-2}$. The cross-section eigenfunctions $\sqrt{2/\delta}\sin \pi l y \delta^{-1} \equiv \nu_l$, $l = 1, 2, \ldots$ span the subspaces $E_l$ and define the corresponding orthogonal projections $P_l$.

# Jump-start modification of the analytic perturbation procedure

Exponential solutions of the homogeneous Schrödinger equation on the leads are

$$\chi_{\pm}^{l}(x,y) = \nu_l(y) e^{\pm i\sqrt{\lambda-[V_\infty+\pi^2\delta^{-2}]}\ x}, \text{ if } \lambda > V_\infty + \pi^2\delta^{-2},$$

in open channels ($l = 1$) and

$$\chi_{\pm}^{l}(x,y) = \nu_l(y) e^{\pm\sqrt{[V_\infty+\pi^2\delta^{-2}-\lambda]}\ x} \text{ if } [V_\infty + \pi^2\delta^{-2} > \lambda]. \quad (6)$$

in closed channels ($l \geq 2$).

# Jump-start modification of the analytic perturbation procedure

The corresponding cross-section subspaces $E_l$ for $\lambda = \Lambda_F$ are called the entrance subspaces of the open and closed channels on the Fermi level:

$$E_{open} = \sum_{[V_\infty + \pi^2 \delta^{-2} < \Lambda_F]} E_l \equiv \sum_{open} E_l,\ E_{closed} = \sum_{[V_\infty + \pi^2 l^2 \delta^{-2} > \Lambda_F]} E_l \equiv \sum_{closed} E_l,$$

and denote by $P_{open} \equiv P_+$, $P_{closed} \equiv P_-$ the corresponding orthogonal projections in the cross-section subspace $E = L_2(0, \delta)$. Introduce also the exponents

$$K_+(\lambda)e = \sqrt{\lambda - [\pi^2 \delta^{-2} + V_\infty]}\ P_+ e,$$

for the open channels and

$$K_-(\lambda)e_- = \sum_{closed} \sqrt{[\pi^2 l^2 \delta^{-2} + V_\infty] - \lambda}\ e_-,\ e_- \in E_- = E \ominus E_+,$$

for the closed channels.

# Jump-start modification of the analytic perturbation procedure

The stationary scattering problem on the junction consists in construction of the scattered waves of the bounded exponential solutions of the Schrödinger equation on the leads

$$\Psi_\omega(x, e, \lambda) = e^{iK_+(\lambda)x}\ e_+ + e^{-iK_+(\lambda)x} S(\lambda)\ e_+ + e^{-K_- x} s(\lambda)\ e, \quad e \in E_+, \tag{7}$$

smoothly matched on $\Gamma$ to the solution $\Psi_\omega(x, \lambda)$ of the corresponding boundary problem on the quantum well:

$$-\Delta\ \Psi_\omega + V\ \Psi_\omega = \lambda \Psi_\omega,$$

$$\Psi_\omega(x, \lambda)\Big|_\Gamma = \Psi_\omega(x, e, \lambda)\Big|_\Gamma,\ \Psi_\omega(x, \lambda)\Big|_{\partial\Omega_{in}\setminus\Gamma} = 0, \tag{8}$$

with the undefined matrix coefficients - the Scattering matrix $S : E_+ \to E_+$ and the amplitude $s : E_+ \to E_-$ of the evanescent waves,- found from the smooth matching condition on on the bottom sections $\Gamma$ of the leads :

# Jump-start modification of the analytic perturbation procedure

$$\Psi_\omega(x,\lambda)\Big|_\Gamma = \Psi_\omega(x,e,\lambda)\Big|_\Gamma,\ \ \frac{\partial\Psi_\omega}{\partial n}(x,\lambda)\Big|_\Gamma = \frac{\partial\Psi_\omega}{\partial n}(x,e,\lambda)\Big|_\Gamma.$$

The smooth matching procedure can be formalized based on the relative intermediate Dirichlet-to-Neumann map $\mathcal{DN}_\Gamma^{\Lambda_F}$ of the quantum well, see [12]. The relative intermediate DN-map is combined of matrix elements of the relative DN-map $\mathcal{DN}_\Gamma$ obtained via restriction of the standard DN-map onto the bottom sections of the leads.

# Jump-start modification of the analytic perturbation procedure

Consider the decomposition of $\mathcal{DN}_\Gamma$ with respect to the orthogonal basis of the cross-section subspaces $E_{open} \oplus E_{closed}$ : $\mathcal{DN}_\Gamma =$

$$\equiv \begin{pmatrix} P_{open}\mathcal{DN}P_{open} & P_{open}\mathcal{DN}P_{closed} \\ P_{closed}\mathcal{DN}P_{open} & P_{closed}\mathcal{DN}P_{closed} \end{pmatrix} \equiv \tag{9}$$

$$\begin{pmatrix} P_+\mathcal{DN}P_+ & P_+\mathcal{DN}P_- \\ P_-\mathcal{DN}P_+ & P_-\mathcal{DN}P_- \end{pmatrix} \begin{pmatrix} \mathcal{DN}_{++} & \mathcal{DN}_{+-} \\ \mathcal{DN}_{-+} & \mathcal{DN}_{--} \end{pmatrix}.$$

# Jump-start modification of the analytic perturbation procedure

Then the intermediate DN-map is f0und as

$$\mathcal{DN}^{\Lambda}_{\Gamma} = \mathcal{DN}_{++} - \mathcal{DN}_{+-}\frac{I}{\mathcal{DN}_{--} + K_{-}}\mathcal{DN}_{-+}. \qquad (10)$$

and the stationary scattering matrix $S$ of the junction is calculated as

$$S(\lambda) = \left[iK_{+} + \mathcal{DN}^{\Lambda}_{\Gamma}\right]^{-1}\left[iK_{+} - \mathcal{DN}^{\Lambda}_{\Gamma}\right]. \qquad (11)$$

The non-stationary scattering matrix for corresponding Schrödinger equation or wave-equation coincides with $S(\lambda)^{+}$ and can be calculated, on the conductivity band of the junction based on wave-operators $W^{in}$, $W^{out}$ via comparison of the perturbed and unperturbed quantum or Lax-Phillips dynamics. In this paper we use the Lax-Phillips dynamics, first of all to prove the chain-rule and obtain a factorization for the scattering matrix with a view of “elimination of dangerous resonances”.

# Chain rule for the scattering matrices and the analytic perturbation problem.

Consider the wave equations $u_{tt} - \Delta u - \left[\pi^2/\delta^2 + q^\infty\right] u = 0$ on the networks $\Omega^0, \Omega^1, \Omega^2$ constituted by the leads $\omega_s$ attached to various vertex domains ( or, more general, *scatterers* ) $\Omega^0_{int}, \Omega^1_{int}, \Omega^2_{int}$. Though we keep the general system of notations, in fact we assume from the very beginning that the vertex domain $\Omega^0_{int}$ is trivial, so that $\Omega^0 = \cup_s \omega_s$, and the scatterer $\Omega_1$ is defined by some inner structure attached to the leads.

# Chain rule for the scattering matrices and the analytic perturbation problem.

Assume that all eigenvalues of the corresponding networks $\Omega^s$, $s = 0, 1, 2$, are embedded into the continuous spectrum of the Laplacian on the leads. Denote by $\mathcal{H}_s$ the energy-normed spaces of the Cauchy data $(u, u_t) = \vec{u}$ on the networks $\mathcal{H}_s = \mathcal{H}$:

$$\| \vec{u} \|^2_{\mathcal{H}} = \frac{1}{2} \int_{\Omega} \left[ |u_t|^2 + |\nabla u|^2 + V(x)|u|^2 - [\pi^2/\delta^2 + q^{\infty}]|u|^2 \right] d\Omega,$$

with the real bounded piecewise continuous potentials $V$ supported by the vertex domain $\Omega_{int}$. The eigenvalues of the situated below the lower threshold in the leads do not contribute to the scattering process, hence will be eliminated from our analysis. Then all energy norms involved are positive.

# Chain rule for the scattering matrices and the analytic perturbation problem.

Let $H_0, H_1, H_2$ be the generators of the corresponding Lax-Phillips unitary groups in the Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$, with equivalent orthogonal incoming and outgoing subspaces $\mathcal{D}_{int}$, $\mathcal{D}_{out}$ : spanned by the entrance spaces of the first spectral channel $\vec{u}(t \pm x, y) = \sqrt{2/\delta} \sin \pi y/\delta \ \vec{u}(t \pm x)$, $0 < x < \infty$ on the leads:

$$e^{iH_s t}\mathcal{D}_{out} \subset \mathcal{D}_{out} \text{ for } t > 0, \text{ and } e^{iH_s t}\mathcal{D}_{in} \subset \mathcal{D}_{in} \text{ for } t < 0.$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

The Adamyan-Arov-Lax-Phillips wave operators are defined, see [21], as

$$W_{r,s}^{in,out} = s - lim_{t \to \pm\infty} e^{-iH_r t} P_{in,out} e^{iH_s t}, \tag{12}$$

and the non-stationary scattering matrix is calculated as

$$S^{LP}(H_r, H_s) = \left[W_{r,s}^{out}\right]^{+} W_{r,s}^{in}. \tag{13}$$

The scattering matrix commutes with the unperturbed operator $H_s$ within the pair $H_r$, $H_s$ and is defined, in the spectral representation of $H_s$, by the multiplication by a matrix $s_{rs}(\lambda)$ depending on the spectral parameter and subsequent integration with the spectral resolution $E_s$ of $H_s$:

$$S^{LP}(H_r, H_s) = \int s_{rs}(\lambda) dE_s(\lambda), \tag{14}$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

The wave operators $W_{r,s}^{in,out}(\lambda)$ connect the parts of the spaces

$$\mathcal{H}_r^{in,out} = \bigvee_{-\infty<t<\infty} e^{iH_r\ t}\mathcal{D}_{in,out},\ \mathcal{H}_s^{in,out} = \bigvee_{-\infty<t<\infty} e^{iH_s\ t}\mathcal{D}_{in,out}$$

$$\mathcal{H}_r^{in} = W_{in}(r,s)\mathcal{H}_s^{in},\ \mathcal{H}_r^{out} = W_{out}(r,s)\mathcal{H}_s^{out},$$

and the corresponding absolutely-continuous parts $H_{r,s}^{in,out}$ of the operators

$$H_r^{in} = H_r\Big|_{\mathcal{H}_r^{in}} = W_{in}(r,s)H_s^{in},\ H_r^{out} = H_r\Big|_{\mathcal{H}_r^{out}} = W_{out}(r,s)H_s^{out}, \tag{15}$$

In our case the scattering matrices are unitary, hence $\mathcal{H}_s^{in} = \mathcal{H}_s^{out}$.

# Chain rule for the scattering matrices and the analytic perturbation problem.

## Theorem

*Consider three unitary groups $U_l$ with Lax-Phillips generators $H_l$, $l = 0, 1, 2$ acting in the spaces $\mathcal{H}_s$ with equivalent incoming and outgoing subspaces $\mathcal{D}_{in,\,out}$ and different co-invariant subspaces $\mathcal{K}_l$ : $\mathcal{H}_l = \mathcal{D}_{in} \oplus \mathcal{K}_s \oplus \mathcal{D}_{out}$. Then the scattering matrix $s_{20}$ admits a factorization*

$$s_{20} = s_{21}\ s_{10}, \tag{16}$$

*where $s_{kl}$ are representations of the corresponding scattering operators in the spectral terms of $U_l$.*

# Chain rule for the scattering matrices and the analytic perturbation problem.

*Proof* We use the wave-operators

$$W_{out}(H_l, H_k) = s - lim_{t\to\infty} U_l(-t) P_{out} U_k(t),$$

$$W_{in}(H_l, H_k) = s - lim_{t\to-\infty} U_l(-t) P_{in} U_k(t),$$

and

$$S(H_l, H_k) = W^+_{out}(H_l, H_k) W_{in}(H_l, H_k) = \int_\sigma s_{lk}(\lambda) dE^k(\lambda).$$

Here, due to absolute continuity and coincidence of the spectra of the generators, $\sigma_l = \sigma_k = sigma$, the spectral measure $dE^k(\lambda)$ is connected with the spectral transformations defined by the scattered waves

$$\mathcal{J}_{kl} = \int dp d\nu \Psi_k(p, \nu) \rangle \langle \Psi_l(p, \nu); k, l = 0, 1, 2.$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

$$dE_l = dpd\nu\Psi_l(p,\nu)\rangle\langle\Psi_l(p,\nu)$$

and

$$\int_\sigma dE^k(\lambda)u = \mathcal{J}^+_{k,k}\ \tilde{u}(\lambda)$$

and the scattering operator is connected with the scattering matrix $s_{lk}(\lambda)$ as

$$S(H_l, H_0) = \mathcal{J}^+_{l0}\ s_{l0}\mathcal{J}_{l0}.$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

Due to the chin-rule for the wave operators:

$$W_{out}(H_2, H_0) = W_{out}(H_2, H_1)\ W_{out}(H_1, H_0),$$

$$W_{in}(H_2, H_0) = W_{in}(H_2, H_1)\ W_{in}(H_1, H_0) \tag{17}$$

we have

$$S((H_2, H_0)) = W^+_{out}(H_1, H_0)\ W^+_{out}(H_2, H_1)\, W_{in}(H_2, H_1)\ W_{in}(H_1, H_0). \tag{18}$$

The central factor of the last product is represented via the scattering operator or via scattering matrix of the pair $H_2, H_1$ as

$$W^+_{out}(H_2, H_1)\, W_{in}(H_2, H_1) = S(H_2, H_1) = \int_\sigma s_{21}(\lambda) dE^1(\lambda). \tag{19}$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

The wave operators intertwine the corresponding parts of generators in $\mathcal{H}^l_{in,out} = \bigvee_{-\infty<t<\infty} U^l D_{in,out}$ and the spectral measures such that for $u_1 \in \mathcal{H}^1_{out} = \bigvee_{-\infty<t<\infty} U^1_t D_{out}$ we have:

$$\int_{\sigma_1} s_{21}(\lambda) dE^2(\lambda) W_{out}(H_2, H_1) u_1 = W_{out}(H_2, H_1) \int_{\sigma} s_{21}(\lambda) dE^1(\lambda) u_1,$$

and, similarly, for $u_2 \in \mathcal{H}^2_{out} = \bigvee_{-\infty<t<\infty} U^2_t D_{out}$,

$$W^+_{out}(H_2, H_1) \int_{\sigma} s_{21}(\lambda) dE^2(\lambda) u_2 = \int_{\sigma} s_{21}(\lambda) dE^1(\lambda) W^+_{out}(H_2, H_1) u_2.$$

# Chain rule for the scattering matrices and the analytic perturbation problem.

$$W^{+}_{out}(H_1, H_0) \int_{\sigma} s_{21}(\lambda) dE^1 W_{in}(H_1, H_0) =$$

$$= \int_{\sigma} s_{21}(\lambda) dE^0 W^{+}_{out}(H_1, H_0) W_{in}(H_1, H_0) = \int_{\sigma} s_{21} s_{10} dE^0_{\lambda}$$

Then :

$$\mathcal{J}_0^{+} \, s_{20} \, \mathcal{J}_0 \, = \mathcal{J}_0^{+} \, s_{21} s_{10} \, \mathcal{J}_0, \text{ or } s_{20}(\lambda) = s_{21}(\lambda) \, s_{10}(\lambda). \quad (20)$$

*The end of the proof*

# Chain rule for the scattering matrices and the analytic perturbation problem.

The Lax-Phillips scattering matrix $s_{20}$ on the first spectral branch coincides with the adjoint of the stationary Schrödinger scattering matrix, calculated above, see (11), for the stationary scattering problem. It admits an analytic continuation onto the upper half-plane of the spectral parameter $p = \sqrt{\lambda - \pi^2/\delta^2 - q^\infty}$ as a bounded analytic function.

# Calculation of the scattering matrix of the thin quantum network.

Consider the problem of calculation of the Lax-Phillips scattering matrix depending on the width $\delta$ of the leads. When the leads become thinner, the squares of zeros $p_r$ and poles $\bar{p}_r$ of the scattering matrix $S_{20}(\lambda, \delta)$ depend analytically on $\delta$ and approach the eigenvalues $p_r^2(\delta) \to \lambda_s$ of the corresponding Dirichlet Schrödinger spectral problem $L_{int}u = \lambda u$ in $\Omega_{int}^2$, from the first and second sheets of the spectral parameter $\lambda = p^2 + \pi^2/\delta^2 + q^{infty}$ respectively. Then the Scattering matrix $S_{20}(\lambda, \delta)$ on a small neighborhood $\rho_s$ of the eigenvalue $\lambda_s$, which does not contain other eigenvalues of $L_{int}$, is not an analytic function of $\delta$, because of zeros of the numerator and the denominator converging to the same point when $\delta \to 0$.

# Calculation of the scattering matrix of the thin quantum network.

Consider the factorization of the scattering matrix $s_{20}$ obtained by splitting off the factor containing the resonance converging to $\lambda_s$:

$$s_{10}(p) = \frac{ipP_+ - \frac{\alpha_s(\delta)Q_s}{\lambda P_+ - \lambda_s} - \beta_s(\lambda,\delta)}{ip + \frac{\alpha_s(\delta)Q_s}{\lambda - \lambda_s} - \beta_s(\lambda,\delta)}, \tag{21}$$

with an hermitian matrix $\beta(\lambda,\delta)$ on the interval $\Delta_\rho = R \cap \rho_s$, containing the remaining details of the DN-map, and the projection $Q_s$ onto the 1d subspace spanned by part $P_+ \frac{\partial \varphi_s}{\partial n}\Gamma$ of the eigenfunction's current in the entrance subspace $E_+$ of the open channel.

# Calculation of the scattering matrix of the thin quantum network.

According to [3] there exist an inner structure $H_1$ such that the factor $s_{10}(p)$ is the scattering matrix of the corresponding pair $H_1, H_0$. Then select an "Intermediate Operator" as $H_1(\delta)$ and notice that, for $\lambda \in \rho$, the factor $s_{21}$ is an analytic function of $\delta$, because the spectral characteristics contained in $\beta_s(\lambda, \delta)$ depend analytically on $\delta$ when $\lambda \in \rho_s$.

# Calculation of the scattering matrix of the thin quantum network.

Hence the calculation of the scattering matrix $s_{20}$ can be done in two steps: the construction of an Intermediate Operator $H_1$ with the scattering matrix emulating the non-analytic (with respect to $\delta$) s-factor $s_{10}$ of $s_{20}$, and then the standard analytic perturbation technique for the pair $H_2, H_1$, which gives the scattering matrix $s_{21}(\lambda)$. The Intermediate Operator $H_1$ plays the role of the “jump-start”, equivalent to one proposed in [4].

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