

On Domination of Nonlinear Wave Interaction in the Energy Balance of Wind-Driven Sea

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The Hasselmann kinetic equation

$$\frac{dN}{dt} = S_{nl} + S_{in} + S_{dis}$$

Three questions:

1. What is exactly N_k ?
2. What is the structure of general solution for equation
3. Which term is the most important in the balance equation

$$S_{nl} = 0?$$

$$S_{nl} + S_{in} + S_{dis} = 0?$$

The answer to the last question is: S_{nl} is the most important term.

Shape of the surface is: $\eta(\vec{r}, t), \quad r = (x, y)$

Fluid is incompressible: $div \vec{V} = 0$

Flow is potential: $V = \nabla \Phi, \quad \Delta \Phi = 0, \quad \Phi = \Phi(\eta, z)$

Potential on the surface is: $\Psi = \Phi(\vec{r}, \eta(\vec{r}, t))$

Ψ_k and η_k are Fourier transforms of Ψ and η

We will use normal variable

$$a_k = \frac{1}{\sqrt{2}} \left\{ \left(\frac{g}{A_k} \right)^{1/4} \eta_k - i \left(\frac{A_k}{g} \right)^{1/4} \Psi_k \right.$$

Here $A_k = k \tanh kH$, H is fluid depth, and

$$\omega_k = \sqrt{g A_k} = \sqrt{g k \tanh kH}$$

The total energy of the fluid, $H = T + U$, can be expanded in Taylor series

$$H = H_0 + H_1 + H_2 + \cdots$$

$$H_0 = \int \omega_k |a_k|^2 dk, \quad H_1 \text{ is cubic in } a_k, a_k^* \quad \text{and } H_2 \text{ is quartic in } a_k, a_k^*.$$

$$H_1/H_0 \sim \mu \qquad H_2/H_0 \sim \mu^2$$

The squared average steepness

$$\mu^2 = \frac{<\eta>^2}{g^2} \frac{\omega_p^4}{}$$

η and Ψ are canonical variables

$$\frac{\partial \eta_k}{\partial t} = \frac{\delta H}{\delta \Psi_k^*}, \qquad \frac{\partial \Psi}{\partial t} = - \frac{\delta H}{\delta \eta_k^*}$$

Statistical description:

$$n_{k\omega} \delta_{k-k'} \delta_{\omega-\omega'} = < a_{k\omega} a_{k',\omega'}^* >$$

Spectrum of elevation:

$$Q_{k\omega} \delta_{k-k'} \delta_{\omega-\omega'} = < \eta_{k\omega} \eta_{k',\omega'}^* >$$

$$Q_{k\omega} = \frac{\omega_k}{2}(n_{k\omega} + n_{-k,-\omega})$$

$$n_k \delta_{k-k'} = < a_k a_{k'}^* > \quad Q_k \delta_{k-k'} = < \eta_k \eta_{k'}^* >$$

Spatial spectra:

$$n_k = \int_{-\infty}^{\infty} n_{k\omega} d\omega$$

$$Q_k = \frac{\omega_k}{2}(n_k + n_{-k}) = \int_{-\infty}^{\infty} Q_{k\omega} d\omega$$

Are the following statements correct?

1. $n_k = \frac{2}{\omega_k} \int_0^{\infty} Q_{k\omega} d\omega$
2. n_k satisfies the Hasselmann equation

Actually these statements are not correct.

In fact:

$$\frac{2}{\omega+k}\int_0^\infty Q_{k\omega}d\omega=N_0(k)+N_1(k)+\cdots$$

$$\frac{N_1(k)}{N_0(k)}\sim \mu^2\left(\frac{k}{k_p}\right)^{3/2}\sim \mu^2\left(\frac{\omega}{\omega_1}\right)^3$$

To find $N_1(k)$ one should perform a canonical transformation excluding cubic term I_1 in the Hamiltonian.

$$a_k=\frac{1}{\sqrt{2}}(q_k+ip_k)$$

Here $q_k,$ p_k are canonical variables

$$\frac{\partial q_k}{\partial t}=\frac{\delta H}{\delta p_k^*}\qquad \frac{\partial p_k}{\partial t}=-\frac{\delta H}{\delta q_k^*}$$

Recall that

$$\frac{\partial a_k}{\partial t}+i\frac{\delta H}{\delta a_k^*}=0$$

We need to perform canonical transformation excluding slave harmonics:

$$p_k, q_k \longrightarrow R_k, \xi_k$$

The generating functional of transformation is

$$S = S[R_k, q_k]$$

and the implicit equations for R_k, ξ_k are

$$p_k = \frac{\delta S}{\delta q_k^*} \qquad \xi_k = \frac{\delta S}{\delta R_k^*}$$

The generating function S has the following form:

$$S = \int R_k q_k^* dk + \frac{1}{2} \int A_{kk_1k_2} q_k q_{k_1} R_{k_2} \delta(k + k_1 + k_2) dk dk_1 dk_2 + \\ + \frac{1}{3} \int B_{kk_1k_2} R_k R_{k_1} R_{k_2} \delta(k + k_1 + k_2) dk dk_1 dk_2$$

The coefficients of generation function:

$$A_{kk_1k_2} = -\frac{1}{4} \left(\frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 + L_1 - L_2}{\omega_0 + \omega_1 - \omega_2} \right) + \frac{1}{4} \left(\frac{L_0 - L_1 - L_2}{\omega_0 - \omega_1 - \omega_2} + \frac{L_1 - L_0 - L_2}{\omega_1 - \omega_0 - \omega_2} \right)$$

$$B_{kk_1k_2} = -\frac{1}{4} \left(\frac{L_0 + L_1 + L_2}{\omega_0 + \omega_1 + \omega_2} + \frac{L_0 - L_1 - L_2}{\omega_0 - \omega_1 - \omega_2} \right) - \frac{1}{4} \left(\frac{L_1 - L_0 - L_2}{\omega_1 - \omega_0 - \omega_2} + \frac{L_2 - L_0 - L_1}{\omega_2 - \omega_0 - \omega_1} \right)$$

Here

$$L_0 = L_{kk_1k_2}, \quad L_1 = L_{k_1k_2}, \quad L_2 = L_{k_2k_1}$$

$$\omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2}$$

The new normal variable b_k

$$b_k = \frac{1}{\sqrt{2}} \left(\left(\frac{g}{A_k} \right)^{1/4} \xi_k - i \left(\frac{A_k}{g} \right)^{1/4} R_k \right)$$

satisfies equation

$$\frac{\partial b_k}{\partial t} + i\omega_k b_k + \frac{i}{2} \int T_{kk_1k_2k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 = 0$$

$$L_0=L_{k k_1 k_2}=-\frac{g^{1/4}A_k^{1/4}}{A_{k_1}^{1/4}A_{k_2}^{1/4}}\left[(k_1 k_2)+A_{k_1}A_{k_2}\right]$$

$$L_1=L_{k_1,k,k_2}\qquad L_2=L_{k_2,k,k_1}$$

$$N_0(k)\,\delta(k-k')=<b_k\,b_{k'}^*>$$

$N_0(k)$ satisfies the Hasselmann equation!

$N_1(k)$ is an important correction for $\omega \geq 4\,\omega_p$

$$\begin{aligned}
N_1(k) = & \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_k - \omega_{k_1} - \omega_{k_2})^2} (N_{k_1} N_{k_2} - N_k N_{k_1} - N_k N_{k_2}) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) dk_1 dk_2 + \\
& \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_{k_1} - \omega_k - \omega_{k_2})^2} (N_{k_1} N_{k_2} + N_k N_{k_1} - N_k N_{k_2}) \delta(\vec{k}_1 - \vec{k} - \vec{k}_2) dk_1 dk_2 + \\
& \frac{1}{2} \int \frac{|V^{(1,2)}(\vec{k}_2, \vec{k}, \vec{k}_1)|^2}{(\omega_{k_2} - \omega_k - \omega_{k_1})^2} (N_{k_1} N_{k_2} + N_k N_{k_2} - N_k N_{k_1}) \delta(\vec{k}_2 - \vec{k} - \vec{k}_1) dk_1 dk_2 + \\
& \frac{1}{2} \int \frac{|V^{(0,3)}(\vec{k}, \vec{k}_1, \vec{k}_2)|^2}{(\omega_k + \omega_{k_1} + \omega_{k_2})^2} (N_{k_1} N_{k_2} + N_k N_{k_1} + N_k N_{k_2}) \delta(\vec{k} + \vec{k}_1 + \vec{k}_2) dk_1 dk_2
\end{aligned}$$

II. Solutions of equation $S_{nl} = 0$

$$S_{nl} = \pi g^2 \int |T_{kk_1, k_2 k_3}|^2 \delta(k + k_1 - k_2 - k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) dk_1 dk_2 dk_3.$$

Thereafter $H = \infty$. Let $|k_3| \ll |k|$, $|k_1| \ll |k|$

$$T_{kk_1 k_2 k_3} \simeq \frac{1}{2} k k_1^2 T_{\theta_1, \theta_3}$$

$$T_{\theta_1, \theta_2} = 2(\cos \theta_1 + \cos \theta_3) - \sin(\theta_1 - \theta_3)(\sin \theta_1 - \sin \theta_3)$$

$$\text{Let } N = N_0(k) + N_1(k), \quad N_1(k) \ll N_0(k)$$

N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij} k^2 \frac{\partial}{\partial k_j} N_1,$$

where D_{ij} is the tensor of diffusion coefficients,

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2 p_i p_j N(\theta, q) N(\theta_3, q)$$

$$\text{If } N_k = k^{-x}, \quad S_{nl} = g^{3/2} k^{-3x+19/2} F(x) \\ F(x) < \infty, \quad \text{if } 5/2 < x < 19/2.$$

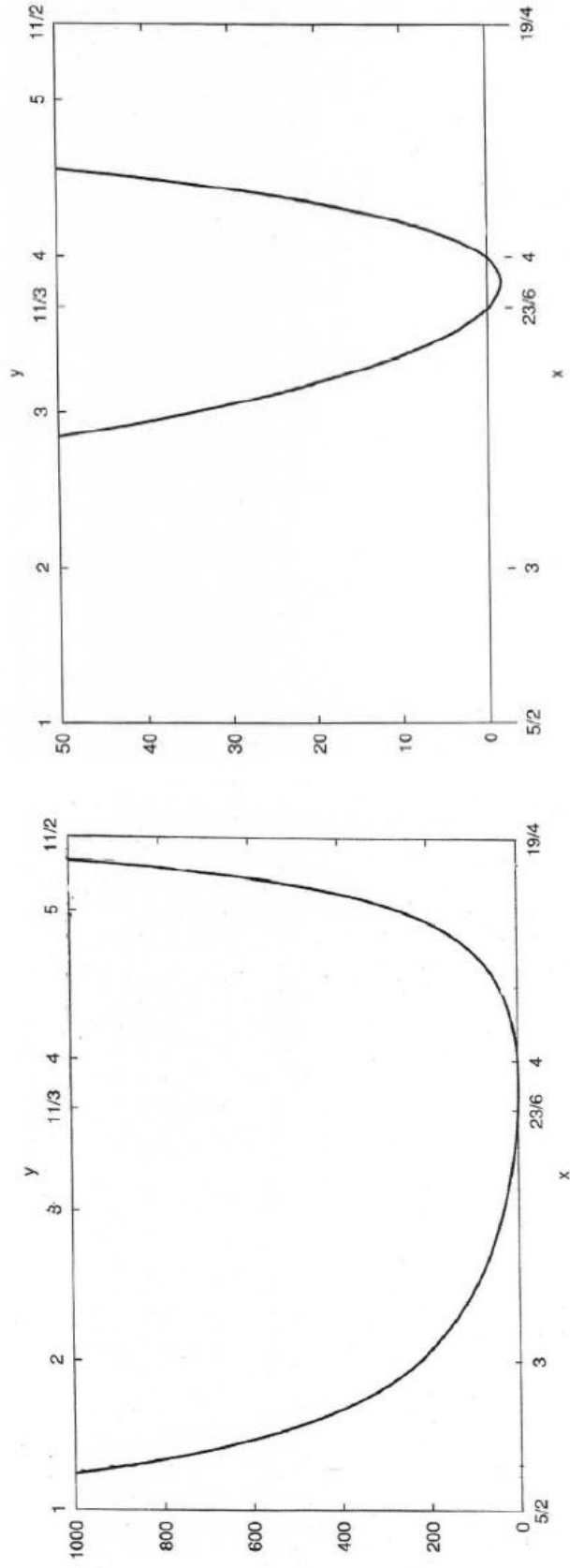


Figure 1: (a) Plot of function $F(x)$. (b) Plot of function $F(x)$: zoom in the vertical direction

On Figure 1 is presented plot of function $F(x)$ for isotropic case calculated numerically. $F(x)$ has two zeros at $x = y_1 = 4$, $x = y_2 = 23/6$.

Corresponding solutions are KZ spectra:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2} \right)^{1/3} \frac{1}{k^4}, \quad N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}} \right)^{1/3} \frac{1}{k^{23/6}}$$

Here $c_p = 0.219$, $c_q = 0.227$ are Kolmogorov constants, P is flux of energy to $k \rightarrow \infty$, and Q is a flux of wave action to $k \rightarrow 0$.

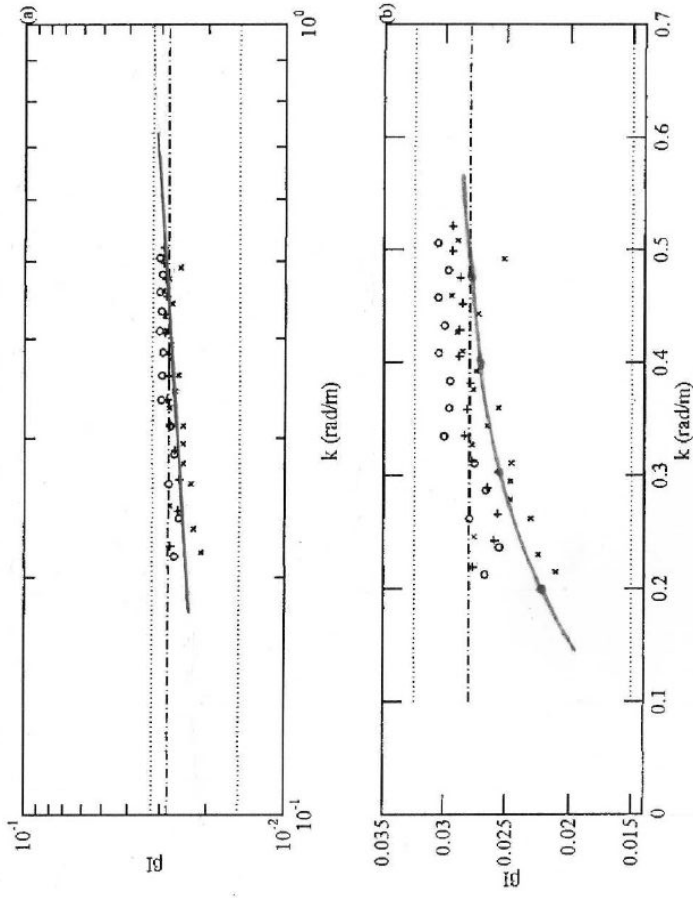


Figure 2: Dimensionless wavenumber spectral coefficient β_i plotted in logarithmic scales (a) and linear scales (b), taken from [1]. Here crosses represent omnidirectional (averaged by angles) spectrum and dots correspond to $\xi(k) = 2\beta_I u_* g^{-0.5} k^{-2.5}$. The solid line on (a) and solid curve on (b) correspond to $\xi(k) \simeq k^{-7/3}$.

A careful study of experimental results shows that in the majority of cases the spectral area right behind the spectral peak can be better approximated by tail $\omega^{-11/3}$ in the frequency spectrum and by tail $k^{-7/3}$ in the spatial spectrum.

- [1] P.A. Hwang, W.C. Wang, E.J. Walsh, W.B. Krabill and R.N. Swift, Airborne measurements of the directional wavenumber spectra of ocean surface waves. Part 1. Spectral slope and dimensionless spectral coefficient. J. Phys. Oceanogr. 30 (2000b), 2753-2767.

A general anisotropic solution

$$N(\omega, \phi) d\omega d\phi = N(\vec{k}) d\vec{k}$$

satisfies equation

$$\frac{\partial N(\omega, \phi)}{\partial t} = S_{nl}(\omega, \phi) = \left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A$$

$$A(\omega, \phi) = \int_0^\infty d\omega' \int_0^{2\pi} d\phi' G(\omega, \omega', \phi - \phi') S_{nl}(\omega', \phi')$$

$G(\omega, \omega', \phi - \phi')$ is a Green function, then

$$\left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) G = \delta(\omega - \omega') \delta(\phi - \phi')$$

$A \rightarrow N$ one to one correspondent. The "diffusion approximation".

$$A = \frac{H_0}{g^4} \omega^{15} N^3, \quad H_0 \simeq 1.83$$

$$S_{nl} = 0 \quad \rightarrow \quad \left(\frac{\partial^2}{\partial \omega^2} + \frac{2}{\omega^2} \frac{\partial^2}{\partial \phi^2} \right) A = 0$$

$$A|_{\omega=\omega_0} = A_0(\phi) = \sum_{n=2}^{\infty} A_n \cos n\phi$$

$$A(\omega, \phi) = A_0 + \frac{A_1}{\omega} \cos \phi + \sum_{n=2}^{\infty} A_n \left(\frac{\omega_0}{\omega} \right)^{-1/2 + \sqrt{1/4 + 4n^2}} \cos n\phi$$

$A_0 = P$ is a flux of energy to $k \rightarrow \infty$

$A_1 = M$ is a flux of momentum to $k \rightarrow \infty$

Other terms describe angular spreading. If there is a flux of action Q from infinity, then

$$A \rightarrow A + \omega Q$$

Dominance of term S_{nl}

The nonlinear interaction term can be presented in the form

$$S_{nl} = F_k - \Gamma_k N_k,$$

where

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2 \delta(k+k_1-k_2-k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) N_{k_1} N_{k_2} N_{k_3} dk_1 dk_2 dk_3$$

and Γ_k , the dissipation rate due to the presence of four-wave processes, is

$$G_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2 \delta(k+k_1-k_2-k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ \times (N_{k_1} N_{k_2} + N_{k_1} N_{k_3} - N_{k_2} N_{k_3}) dk_1 dk_2 dk_3$$

$$S_{in} = \gamma_{in}(k) N_k \quad S_{dis} = -\gamma_{dis}(k) N_k$$

$$\gamma_k = \gamma_{in}(k) - \gamma_{dis}(k)$$

$$S_{nl} + \gamma(k) N_k = 0 \quad N_k = \frac{F_k}{\Gamma_k - \gamma_k}$$

A rough estimate

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k} k^{10} N_k^2$$

Γ_k, F diverge on KZ spectra! As a result for the "mature sea", for the narrow in angle spectrum

$$N_k \simeq \frac{3}{2} \frac{E}{\sqrt{g}} \frac{k_p^{3/2}}{k^4} \theta(k - k_p)$$

we get equation

$$\Gamma_\omega = 36 \pi \omega \left(\frac{\omega}{\omega_p} \right)^3 \mu_p^4 \cos^2 \theta$$

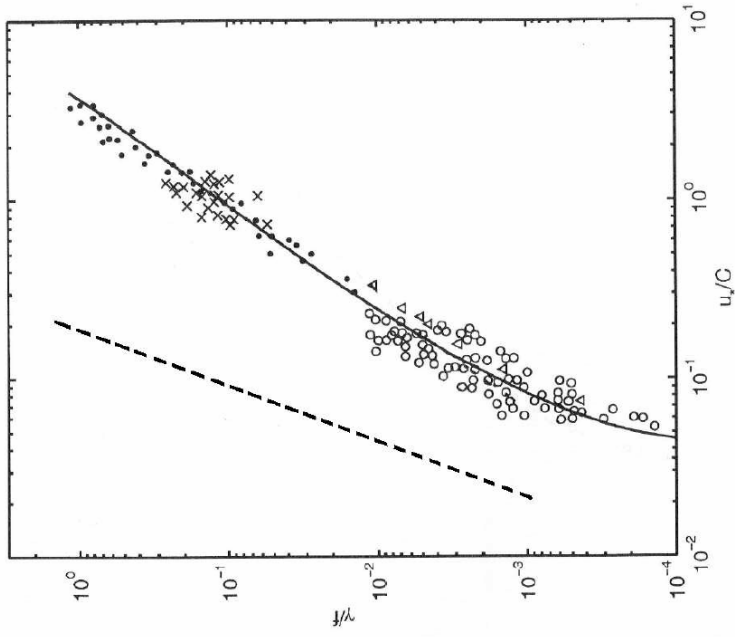
that includes a huge enhancing factor $36\pi \simeq 113.04$. For the very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4} \omega \left(\frac{\omega}{\omega_p} \right)^3 \cos^2 \theta$$

For isotropic spectra

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2} \omega \left(\frac{\omega}{\omega_p} \right)^3 \mu_p^4$$

the enhancing factor $45\pi/2 \simeq 70.65$.



$$\Gamma_{\omega} = 36 \pi \omega \left(\frac{\omega}{\omega_p} \right)^3 \mu_p^4 \cos^2 \theta \quad (1)$$

Figure 3: Comparison of experimental data for the wind-induced growth rate $2\pi \gamma_{in}(\omega)/\omega$ taken from [2] and the damping due to four-wave interactions $2\pi \Gamma(\omega)/\omega$, calculated for narrow in angle spectrum at $\mu \simeq 0.05$ using Eq. (1) (dashed line)

[2] S.J. Komen, L. Cavaleri, M. Donelan, K. Hasselmann, S. Hasselmann, P.A.E.M. Janssen. Dynamics and Modelling of Ocean Waves. Cambridge University Press, 1994.

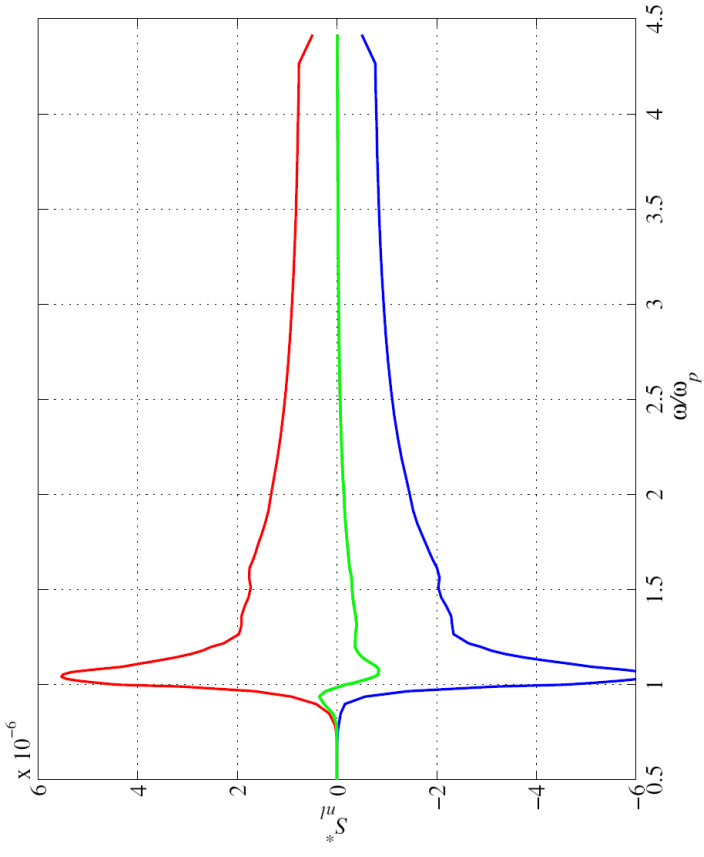


Figure 4: Split of nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k N_k$ (lower curve)

For the nonlinear interaction term $S_{nl} = F_k - \Gamma_k N_k$ the magnitudes of constituents F_k and $\Gamma_k N_k$ essentially exceed their difference. They are one order higher than the magnitude of S_{nl} !

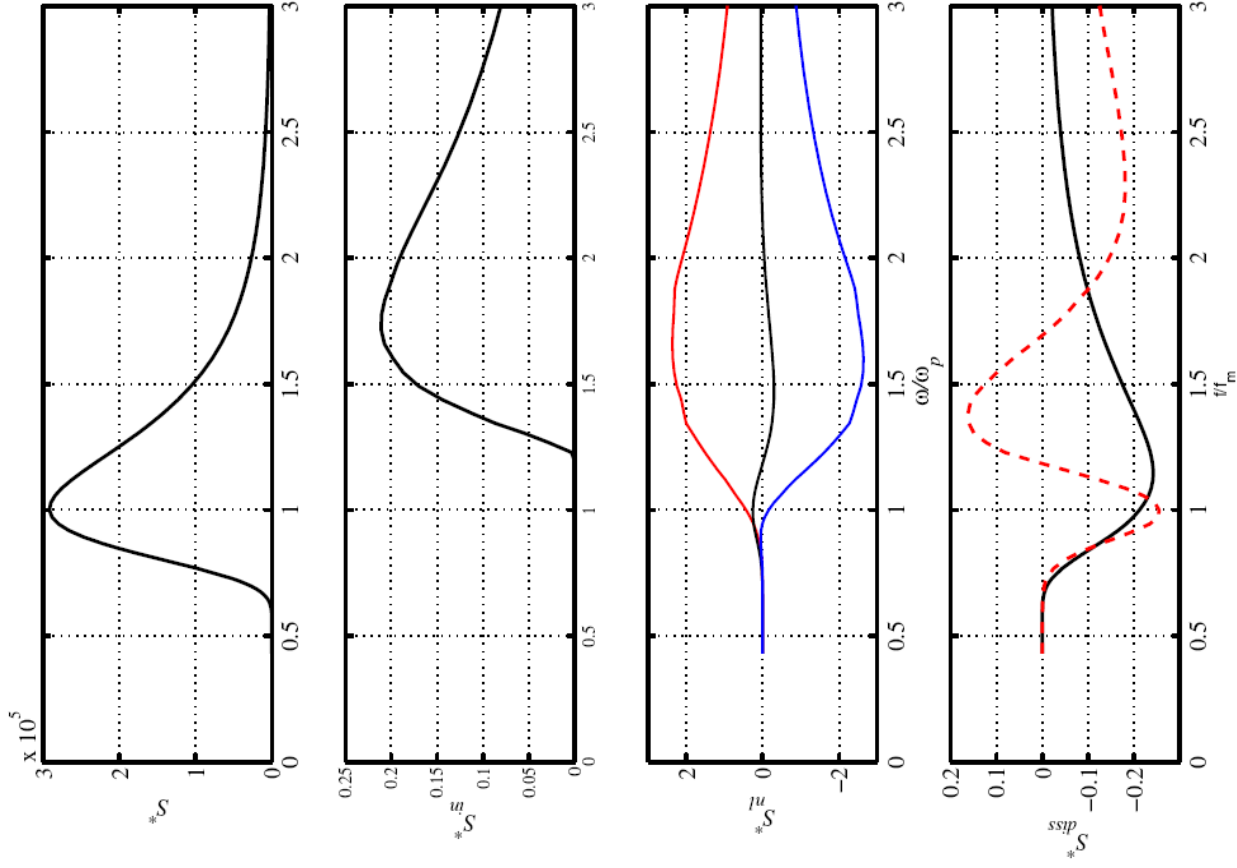


Figure 5: Plots from the
article [3] and the split of S_{nl}
(panel 3)

[3]Komen, G.J., S. Hasselmann S. and K. Hasselmann,
On the existence of a fully developed wind-sea spectrum. J.
Phys. Oceanogr. 14 (1984), 1271-1285.

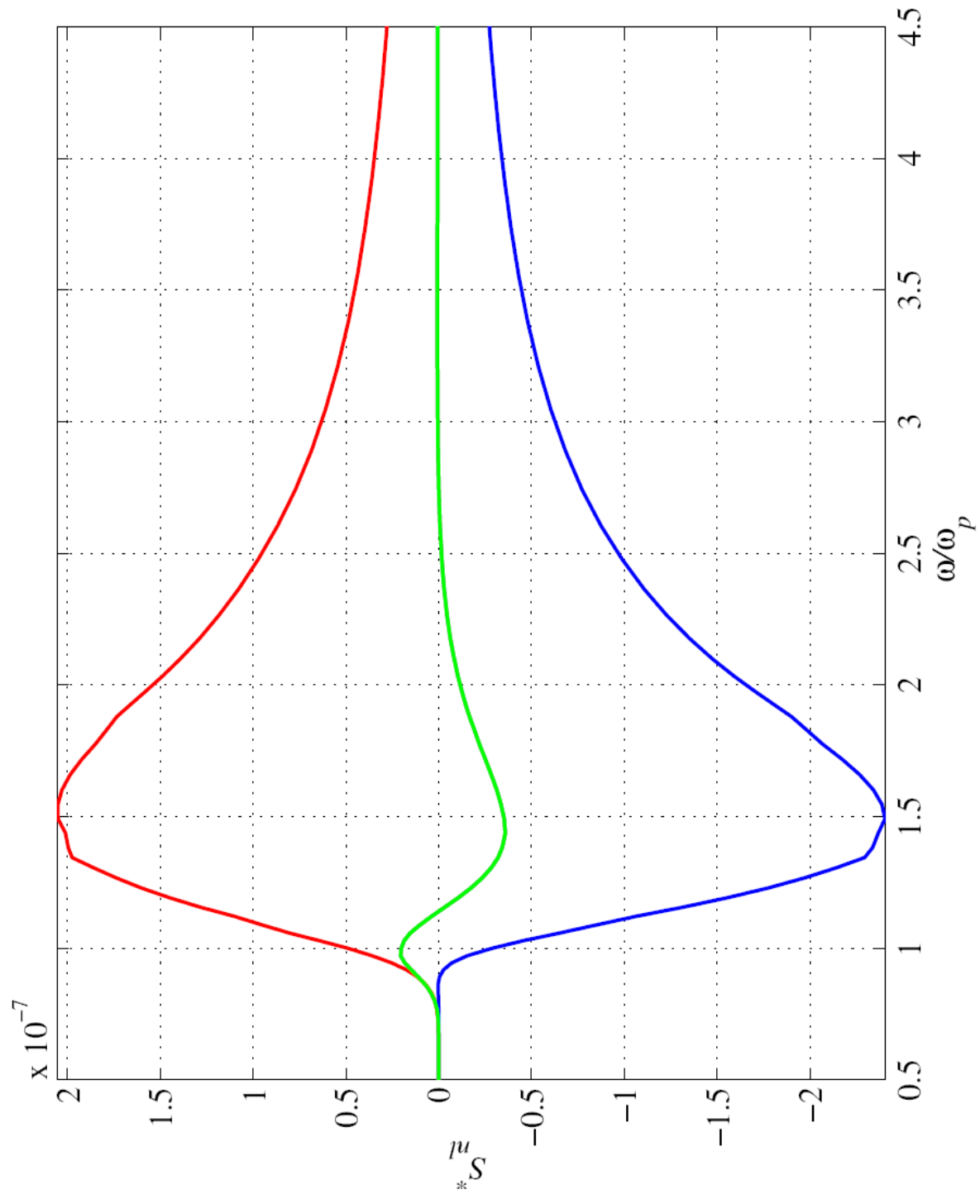


Figure 6: Zoom of panel 3

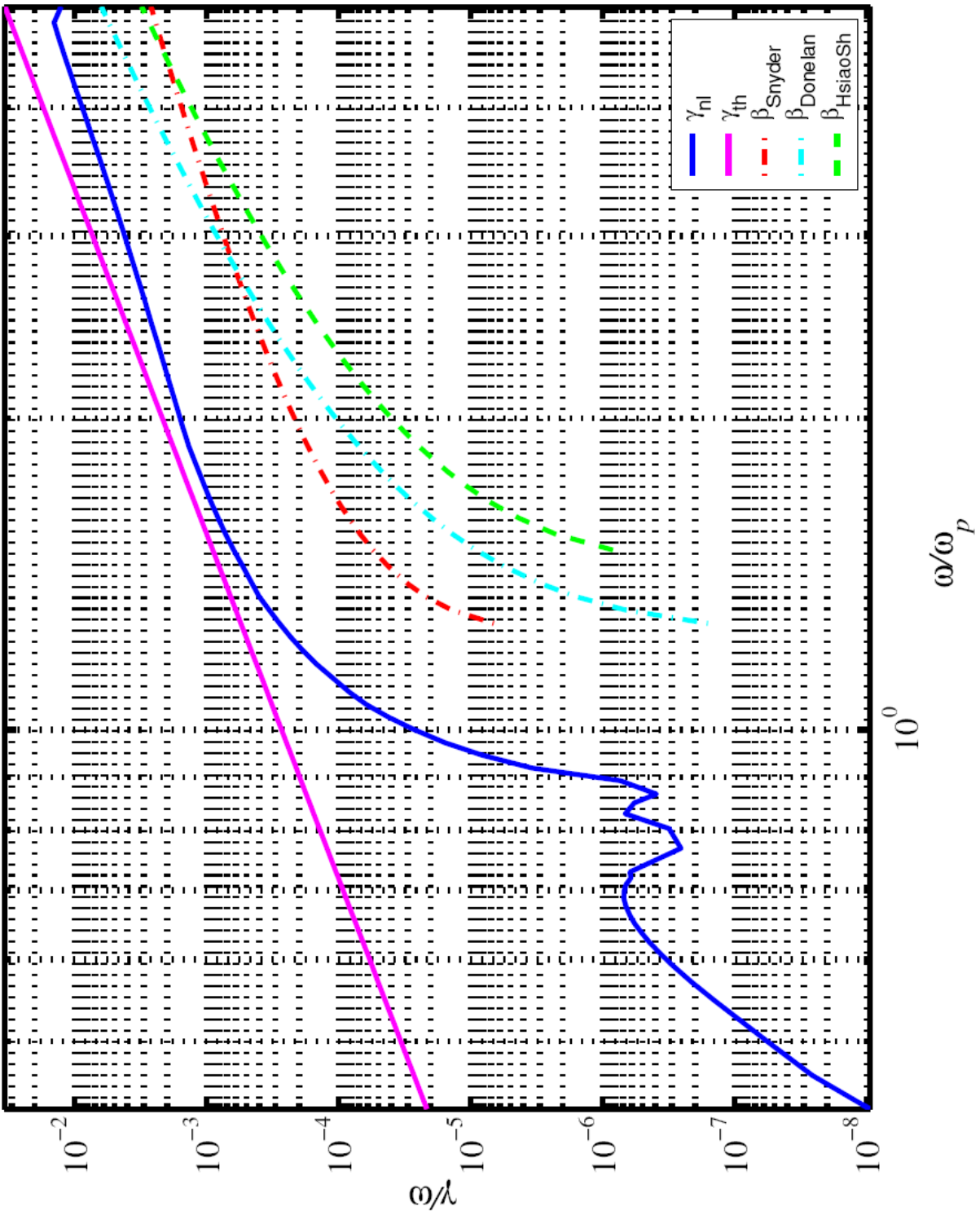


Figure 7: Γ_k compared with different models for S_{in}