Turbulent transfer of energy by radiating pulses

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- The Majda-McLaughlin-Tabak model
- Energy transfer by radiating pulses
- Instability of wave turbulence

Majda-McLaughlin-Tabak (MMT) model

for weakly nonlinear waves

$$\left(i\frac{\partial}{\partial t} - \mathcal{L}\right)\psi(x,t) \quad = \quad \lambda\psi(x,t)|\psi(x,t)|^2$$

- complex wave amplitude $\psi(x,t)$
- linear operator $\mathcal{L}\exp(ikx) = \omega_k \exp(ikx)$,
- dispersion $\omega_k = \sqrt{|k|}$.

(Fourier modes $a_k = \int_{-L/2}^{L/2} \psi(x,t)\exp(-ikx)dx/\sqrt{2\pi}$
Large system size L with periodic boundary conditions)

A.J. Majda, D.W. McLaughlin, E.G. Tabak, J. Nonlinear Sci. 6, 9 (1997),

Conserved quantities of the MMT equation

- Hamiltonian or 'energy'

$$E = \sum_k \omega_k |a_k|^2 + (\lambda/2) \int_{-L/2}^{L/2} |\psi|^4 dx$$

- waveaction

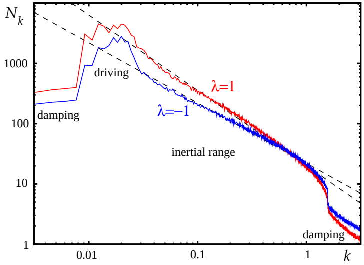
$$N = \sum_k |a_k|^2$$

- momentum

$$P = \sum_k k |a_k|^2$$

Damped and driven MMT equation

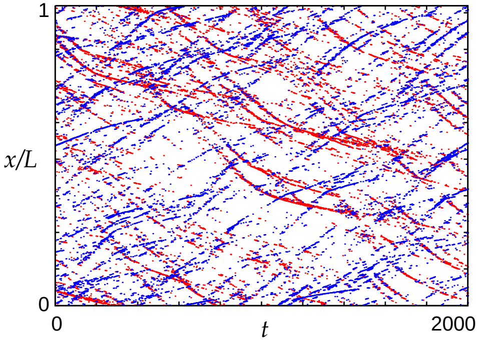
- Driving force
  at moderate wavenumbers
- Damping at high
  and at low wavenumbers
- $N_k = \langle |a_k|^2 \rangle$



- $\lambda = -1$: Kolmogorov-Zakharov spectrum $N_k \sim k^{-1}$
  $\longrightarrow$ wave turbulence
- $\lambda = 1$: Steeper spectrum $N_k \sim k^{-1.25}$
  $\longrightarrow$ unknown mechanism of turbulence

Formation of coherent structures for $\lambda = 1$:

A gas of solitary waves

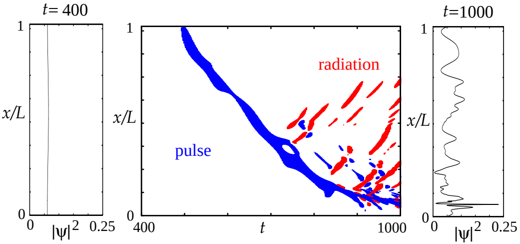


Pattern of solitary waves ('pulses') with high positive or negative momenta

Benjamin-Feir instability for $\lambda = 1$

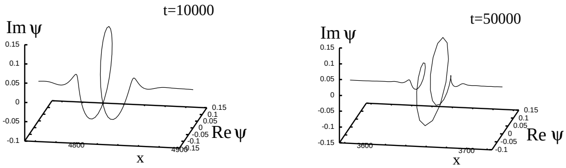
- Monochromatic wave $\psi = (\psi_0 + \delta a)\exp(ik_m x)$
- Modulation $\delta a = \delta a_+ \exp(iqx) + \delta a_- \exp(-iqx)$
- Instability with the largest eigenvalue for $q \approx k_m$.

Pulses emerge from the Benjamin-Feir instability



- The pulse-speed decays
- The pulse emits radiation

A radiating pulse evolves in time



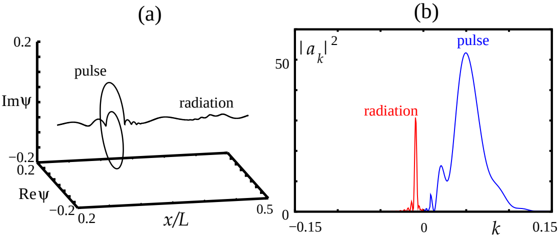
- Shape of a pulse

$$\psi^{(f)} = q\sqrt{\omega_m}{k_m}^{-1} f(\theta)\exp(i\alpha)\exp(i\Omega t)$$

with $\theta_x = q$, $\theta_t = -qv$, $\alpha_x = k_m$, $\alpha_t = k_m v$

- The pulse narrows in real space
- The number of loops increases

Radiation: Resonant driving of linear waves



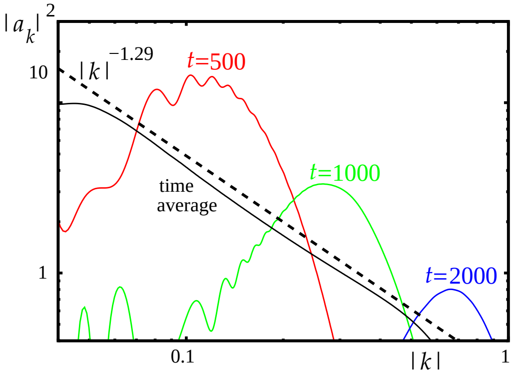
linear wave: driving force by the pulse:

$$i\dot{a}_k - \omega_k a_k \quad = \quad |T_k| \exp(-i\Lambda_k t)$$

with

$$T_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^{(f)} |\psi^{(f)}|^2 \exp(-ikx) dx$$

Time-average of an evolving pulse is MMT-like



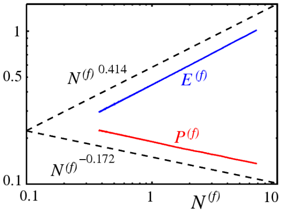
- Pulse in k-space at three different times
- Time-average spectrum $\langle |a_k^{(f)}|^2 \rangle \sim k^{-1.29}$

Evolution of energy and momentum

- Balance of energy $E^{(f)}$ and momentum $P^{(f)}$ of the pulse and the radiation yields

$$\left(\frac{dE^{(f)}(N^{(f)}, P^{(f)}(N^{(f)}))}{dN^{(f)}}\right)^2 = -\frac{dP^{(f)}(N^{(f)})}{dN^{(f)}}$$

- $N^{(f)}$ decays in time
- $E^{(f)} \approx \sqrt{N^{(f)} P^{(f)}} \sim {N^{(f)}}^{\sqrt{2}-1}$ decays
- $P^{(f)} \sim {N^{(f)}}^{\sqrt{8}-3}$ increases
- width of pulse $q \sim k_m^{(3\sqrt{2}-5)/(2\sqrt{2}-2)}$ (k_m: maximum pulse-amplitude)



- Radiation driven by a pulse:

$$i\dot{a}_k - \omega_k a_k = |T_k| \exp(-i\Lambda_k t)$$

- Time-dependent pulse frequency with linear chirp approximation $\Lambda_k(t) \approx \omega_k + \dot{\Lambda}_k t$
- Amplitude of radiation

$$|a_k|^2 \sim T_k^2 \sqrt{|k_m|}/\dot{k}_m$$

after the driving frequency $\Lambda_k(t)$ has moved through resonance

The spectrum of the pulses

- Driving force $T_k \sim q^2 k_m^{-9/4}$
- Speed of a pulse in k-space: $\dot{k}_m \sim q^3 k_m^{-3/2}$
- Wave action of a pulse $N^{(f)} \sim q k_m^{-3/2}$
- Spectrum:

$$
\begin{aligned}
\langle |a^{(f)}_{k=k_m}|^2 \rangle &\sim N^{(f)}/\dot{k}_m \\
&\sim q^{-2} \\
&\sim \sim k_m^{(3\sqrt{2}-5)/(2-\sqrt{2})} \sim k_m^{-1.29}
\end{aligned}
$$

Envelope equation for wave turbulence

- Slow spatial variations of the waveaction $N(k, x, t)$
- Envelope governed by a Vlasov equation

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial k}\frac{\partial N}{\partial x} - \frac{\partial \tilde{\omega}}{\partial x}\frac{\partial N}{\partial k} = 0$$

- Nonlinear by renormalized frequency

$$\tilde{\omega}(k, x, t) = \omega_k + 2\lambda \int N(p, x, t)dp$$

- Linearization $N(k,x,t) = N_0(k) + \Delta N(k,x,t)$
- Modulation $\Delta N(k,x,t) = a(k)\exp(iKx - i\Omega t)$

$$\Delta N(k,x,t) = \frac{2\lambda}{\omega'_k - \Omega/K}\frac{dN_0}{dk}\int \Delta N(p,x,t)dp.$$

Examples:

- monochromatic wave $N_0(k) = A_0\delta(k-k_1)$:
  $\Omega = K(\omega'_{k_1} \pm \sqrt{2\lambda A_0 \omega''_{k_1}})$.
  Benjamin-Feir instability for $\lambda\omega''_{k_1} < 0$
- KZ spectrum $N_0(k) \sim k^{-1}$:
  $\lambda > 0$: modulationally unstable $\rightarrow$ MMT spectrum
  $\lambda < 0$: unstable at breakdown $\rightarrow$ collapses

Conclusions

Modulational instability of wave turbulence

- Description by a Vlasov equation

Turbulent transfer of energy by radiating pulses:

- Direct energy cascade by evolving pulses;
  the specrum follows from the time-average of the pulses
- Inverse cascade of wave action by radiation

B.R., A.C. Newell, V.E. Zakharov, PRL 103, 074502 (2009); preprint