Some Singular and Nonsingular solutions to the integrable PDEs

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Talk delivered at the conference "Frontiers in Nonlinear waves"
in honor of Vladimir Zakharov 70-th anniversary
Tucson, Arizona, US

March 25, 2010

Main Results

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1. Singular rational solutions to the KP-II equation.
    - Wronskian dressing formula for linear evolution PDE's.
    - Wronskian dressing formula for the KP-I and KP-II equations
    - First applications to the KP-II equation
    - Some remarks on the rational solutions of the KP-II equation
1. Rational and Multi-positons solutions to the KdV equation
    - Generalized Wronskian dressing formula
    - Rational Solutions to the KdV equation
    - Multi-positons solutions and multi positon-soliton solutions
        - Positon Solution of the KdV Equation
        - Soliton-positon solution of the KdV equation

Résumé

We present some comments concerning the construction of the rational solutions to the integrable PDEs taking as basic example KP-I, KP-II and KdV equation. We also briefly describe the so-called positon and multi-positons solutions to the equation and formulate some unsolved problems.

At this conference we can celebrate not only Vladimir Zakharov's 70-th anniversary but also the Kadomtsev-Petviashvily-equation 40-th birthday. Zakharov-Shabat pioneering work of 1974 (FA **8** n.4. ,43-52) was a first and most important cornerstone at the base of the beautiful building of the further theories relevant to Zakharov-Shabat hierarchy and its generalizations. Their work was already containing a systematic derivation of multi-lines solitons solutions, the solutions to KP-I and KP-II equations depending on any number of functional parameters and many other things, (some of them in a form at that time hidden even from the authors). That's why I decided to present here my own views on several related topics starting from our joint work with Volodya Zakharov which may be considered as one the pull-outs of those hidden things).

Physics Letters A v. 67 n. 3 ,p. 205-206 (Zakharov ,Manakov ,Bordag,Its,Matveev) KP-I equation reads

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy}$$

For every n it has a family of smooth rational solutions depending on $2m = N$ complex parameters :

$$\nu_j, \xi_j, \quad \nu_{j+m} = -\nu_j, \quad \xi_{m+j} = \bar{\xi}_j, \quad j = 1 \ldots m$$

this solution reads :

$$u = 2\,\partial_x^2 \log\det A,$$

$$A_{pq} = \delta_{pq}(x - i\nu_p y - \xi_p - 3\nu_p^2 t) + \frac{2(1-\delta_{pq})}{\nu_p - \nu_q}$$

When $t \to \infty$ this solution asymptotically splits into a sum of individual lumps with velocity vectors $v_p = (3|\nu_p|^2, -6\Im\nu_p)$, with no phase shifts caused by the lumps interactions.

In a sequel $W(f_1, f_2, \ldots, f_n)$ denotes a Wronskian determinant of m functions $f_j(x)$ i.e.

$$W(f_1, f_2, \ldots, f_n) := \det ||\partial_x^{j-1} f_k(x)||; \quad j, k = 1, \ldots, n.$$

Suppose $f_1, \ldots f_n, f$ are $n+1$ linearly independent solutions of some PDE of the form

$$f_t = \sum_{j=0}^{m} u_j(x,t) f^{(j)}, \quad f_t := \partial_t f, \quad , f^{(j)} := \partial_x^j f.$$

Than the following statement holds (Matveev 1979) [3, 4])

Theorem

The function

$$\psi_n(x,t) := \frac{W(f_1,\ldots f_n, f)}{W(f_1,\ldots f_n)}$$

satisfies the PDE of the same form with the coefficients $\tilde{u}_j$ expressed by means of the functions $f_1,\ldots f_n$. In particular, in the case $u_m = 1, u_{m-1} = 0$, the coefficients $\tilde{u}_m = 1, \tilde{u}_{m-1} = 0$ and

$$\tilde{u}_{m-2} = u_{m-2} + m\partial_x^2 \log W(f_1,\ldots f_n).$$

The function $\psi_n(x,t)$ is called the n-fold Darboux transformation of f.

Compatibility of the linear system

$$\alpha f_y = f_{xx} + uf$$
$$f_t = f_{xxx} + \frac{3}{2} u\, f_x + vf$$

implies that $u(x, y, t)$ is a solution to the KP-I equation :

$$\partial_x (4u_t + 6uu_x + u_{xxx}) = 3u_{yy},$$

when $\alpha = i$, and to KP-II equation :

$$\partial_x (4u_t + 6uu_x + u_{xxx}) = -3u_{yy}.$$

when $\alpha = i$. Suppose that $f_1, \ldots f_n, f$ are the linearly independent solutions of the system above and u satisfies the KP-I or KP-II equation.

The following statement was first proved and applied by the present speaker [3] in January 1979,(Lett. Math. Phys **3**, p.213-216)

Theorem

The formula

$$\tilde{u} := u + 2\partial_x^2 \ln W(f_1, \dots f_n),$$

describes a new solution to KP-I or respectively KP-II equation parametrized by the choice of $f_1, \dots f_n$.

$$f(k,x,y,t) = e^{kx+k^2y+k^3t},$$

is obviously a solution to the Zakharov-Shabat linear system above with $u = 0, \alpha = 1$. The following choice f_j in the Wronskian formulas leads to a large class of solutions to the KP-II equation depending on n "arbitrary" chosen functional parameters $\rho_j(k)$:

$$f_j = \int_{\Delta_j} \rho_j(k_j) d\, k_j.$$

Assuming that all functions $\rho_j(k)$ have a compact support we can represent the related Wronskian τ-function as a multiple integral [8, 5], (V.M., M.S 1977) :

$$W = \int \dots \int e^{\sum_{j=1}^{n} k_j x + k_j^2 y + k_j^3 t} \prod_{j<p} (k_p - k_j) \prod_{j=1}^{n} \rho_j(k_j) d\, k_j.$$

The last observation follows from the fact that any Wronskian determinant is a linear function of its columns. Assuming that $k, \rho_j(k)$ are real we get the real valued solutions to the KP-II equation. Imposing further restrictions on the densities, we get also the nonsingular solutions to the KP-II equation. For instance, it is enough to assume that all densities $\rho_j(k) \geq 0$ and do not vanish identically , all theirs supports are non intersecting and are situated on a positive semi-axis. For other sufficient conditions see my works with M. Salle [8, 5]. The constructed class of the solutions to the KP equation is extremely rich. Still is not studied well enough. In particular, a very challenging question is how it behaves when n-goes to infinity ? Contrary to the similar questions arising in the theory of the matrix models here we have instead of square of Vandermond determinant its first degree which makes the study much more involved.

The so-called multiline-solitons solutions, first constructed by Zakharov and Shabat in 1974 correspond in this picture to take the densities in form of $delta-functions$ concentrated at the different points.

I have no intention to discuss them here.

One general remark concerning the "1-fold " Darboux dressing is that it is already provide an infinite-dimensional family of solutions to the KP-equation due to the fact that that they have the form :

$$u = 2\partial_x^2 \log f, \quad \text{where} \quad f_y = f_{xx}, \; f_t = f_{xxx}.$$

This leads in particular to an infinite family of polynomial τ-functions expressed by means of linear combinations of Schur polynomials aka Bell polynomials or Faa di Bruno polynomials.

This remarks follows my article (LMP n. 3 , 512-525 , 1979) were Schur polynomials were first employed in the context of description of the rational solutions to the KP-II equation, although there I called them Faa di Bruno polynomials which is also a reasonable name for them.
One family of rational solutions to the KP-II equations is obtained from the following choice of f_j in the Wronskian formula above :

$$f_j = (\partial_k + g(k)) \exp(kx + k^2 y + k^3 t)|_{k=k_j}, \quad j = 1 \ldots , n$$

Here $g(k)$ is an arbitrary smooth and real valued function of k. The related elements of the Wronskian are linear functions of x, y and t modulo exponential factor. The exponentials obviously can be pulled out of the Wronskian and disappear after taking the second derivative of logarithm of Wronskian. This is exactly the the same family of the solutions as constructed by Krichever [7] but his derivation employing

The Wronskian formula above provides the same result via one-line calculation. In [7] these solutions were called the solutions of general position.
The work by Krichever was containing a remarkable result saying that all decreasing rational solutions to the KP-I are of the form

$$u = -2 \sum_{j=1}^{n} \frac{1}{(x - x_j)^2},$$

and the dependence of the poles $x_j(y, t)$ on y realize all the trajectories of Calogero-Moser system of n-particles on the line with Hamiltonian

$$H(x, p) = \sum_{j=1}^{n} \frac{p_j^2}{2} + \sum_{j<m} \frac{2}{(x_j - x_m)^2}.$$

The solutions of the general position are characterized by asymptotically free motions of the related particles. Below, following my work (LMP ,textbf3, 505-512 (1979)) we present the construction of a very large variety of solutions to the KP-II equation corresponding to the separatrix trajectories of the same system. For these solutions $x_j(y, t)$ are not asymptotically linear function of y when $y \to \pm\infty$ in terms of generalized Schur polynomials. The text of this article was not included in our book "Darboux transformations and solitons and probably by this reason is cited not enough.

In the same article the construction of the rational solutions for the whole KP hierarchy containing an infinite number of time variables in terms of the Schur polynomials was also obtained. In this sense the typical claim that the connection of the Schur polynomials with KP hierarchy was first discovered by Sato [**?**] in 1981,(i.e. two years later with respect to my article), seems to be not very correct.

Consider the system of m linear PDEs :

$$\partial_{t_1} f = \partial_{t_j}^j f, \quad j = 2, \ldots, m; \quad t_1 \equiv x, t_2 \equiv y, t_3 \equiv t.$$

As a generating solution for this system we can take e^σ where

$$\sigma := \sum_{j=1}^{m} k^j t_j$$

Since the operators

$$\partial_k, \partial_{t_j}, \partial_{t_l}$$

commute , an operator

$$R(j_1, \ldots, j_{m+1}) := \sum a(j_1, \ldots, j_{m+1}) \partial_{t_1}^{j_1} \ldots \partial_{t_m}^{j_m} \partial_k^{j_{m+1}},$$

where $a(j_1, \ldots, j_{m+1})$ are any constant coefficients, maps e^σ to another solution $G\, e^\sigma$ of the same system where G is a polynomial with respect to k and all variables t_j.

Substituting $f_j := Ge^{\sigma}|_{k=k_j}$ into the Wronskian formula we get much larger class of the rational solutions to the KP-II equation. In particular all the functions $P_n e^{\sigma}$ satisfies the system at the previous slide for all values of k, where

$$P_n(k, t_1, \ldots, t_n) := e^{-\sigma}\partial_k^n e^{\sigma}$$

is a Schur polynomial. For all values of k, $P_n(t_1, \ldots, t_n)$ is a polynomial $tau - function$ of the KP-II equation depending on x, y, t and on $n-2$ additional parameters $k, t_4, \ldots, t_n$. Any finite linear combination $\sum c_j P_j(k, t_1, \ldots, t_j)$ is again a polynomial τ-function of the KP-II equation. The polynomials $P_n(k, t_1, \ldots, t_n)$ can be computed using the formula :

$$P_n = \left(\partial_k + \sum_{j=1}^{m} jk^{j-1} t_j\right)^n \cdot 1, \quad P_0 = 1.$$

It particular for $k = 0$, $m \geq n$ we have

$$P_n(t_1, \ldots, t_n) = P_n(t_1, \ldots, t_m),$$

and a following determinant representation for $P_n(t_1, \ldots, t_n)$ holds (Matveev 1979, McDonald 1979) :

$$\begin{vmatrix} t_1 & -1 & 0 & 0 & 0 & \ldots & 0 \\ 2t_2 & t_1 & -2 & 0 & 0 & \ldots & 0 \\ 3t_3 & 2t_2 & t_1 & -3 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ddots & 0 \\ (n-1)t_{n-1} & \ldots & \ldots & \ldots & \ldots & t_1 & 1-n \\ nt_n & (n-1)t_{n-1} & \ldots & \ldots & 3t_3 & 2t_2 & t_1 \end{vmatrix}$$

The polynomials P_n also obey the relations :

$$P_n(t_1,\ldots,t_j,0,\ldots,0)=P_n(t_1,\ldots,t_j),$$

$$P_n(qt_1,\ldots,q^m t_m)=q^m P_n(t_1,\ldots,t_m),$$

$$\partial_{t_1}P_n(t_1,\ldots,t_m)=nP_{n-1}(t_1,\ldots,t_m)$$

With respect to the KP-II equation, the dependence on the parameters $t_1,\ldots,t_m$ represent the action of the higher Zakharov-Shabat hierarchy flows transforming the Schur polynomial τ-functions to the new polynomial τ-functions of the similar structure. First five $P_n(x,y,t)$ are listed below. $P_n(x,y,t)$ are listed below :

$$P_0=1,\quad P_1=x,\quad P_3=x^3+6xy+6t,$$

$$P_4=x^4+12x^2y+24tx+12y^2$$

$$P_5=x^5+20x^3y+60x^2t+60xy^2+120yt.$$

$$u_t = 6uu_x - u_{xxx}, \tag{1}$$

$$\begin{gathered} -f_{xx} + uf = k^2 f, \\ f_t = -4f_{xxx} + 6uf_x + 3u_x f, \end{gathered} \tag{2}$$

Assume that $f_j(k_j, x, t), \quad j = 1, \ldots, n$ are the linearly independent solutions of the Lax system for $k = k_j$, smoothly depending on k_j Denote $f_j^{(l)} := \partial_{k_j}^l f_j, \quad l = 1, \ldots, l_j$ their derivatives. Compose two Wronskian determinants

$$W_1 = W(f_1, \ldots, f_1^{(l_1)}, f_2, \ldots, f_2^{(l_2)}, \ldots, f_n, \ldots, f_n^{(l_n)}, f),$$

$$W_2 = W(f_1, \ldots, f_1^{(l_1)}, f_2, \ldots, f_2^{(l_2)}, \ldots, f_n, \ldots, f_n^{(l_n)}).$$

Then following theorem holds.

Theorem

(V.B.Matveev Phys. Lett.A (1992), 205-208)
For any solution $u(x,t)$ of the KdV equation

$$\tilde{u} := u - 2\partial_x^2 \log W_2,$$

is a new solution to the KdV equation. The related solution to the Lax linear system written for $\tilde{u}$ is given by the formula

$$\psi = \frac{W_1}{W_2}$$

The proof is obtained taking an appropriate limit in standard Wronskian dressing formula formed by

$$n + \sum_{j=1}^{n} l_j$$

linearly independent solutions of the Lax system.

Suppose that

$$u = 0, \quad , \operatorname{Im} k_j = 0, \forall j, \quad j = 1, \dots, n$$

and take in the formulas of the previous theorem

$$f_j(k_j, x, t) = e^{kx - 4k^3 t + P_j(k)}|_{k=k_j},$$

where $P_j(k)$ are arbitrary real polynomials of order $\leq l_j$. The obtained rational solutions depend on $2n + \sum_{j=1}^{n} l_j$ real parameters. In particular, in one-point case, generated by the single exponential we recover the family of rational solutions constructed in a different but longer way by Krichever (1978,1979) [7]. Our approach again provides a 1-line derivation of the larger class of solutions.

Here we show how to get from the generalized wronskian formulae some long-range oscillating solutions introduced by the present speaker in 1992, (Phys. Lett.A v. 166, 209-212),[14]. The author proposed calling them positons since, in spectral sense, they are connected with Wigner-von-Neumann resonances : spectral singularities imbedded in the positive continuous spectrum of the Schrödinger operator (for a more detailed study of their properties and interactions with solitons, see [15] or [16]).

This solution reads :

$$
\begin{aligned}
&u := -2\partial_x^2 \log W(p, p_k) = \\
&-2\partial_x^2 \log(2kg - \sin 2p) = \\
&= \frac{32k_1^2(\sin T - kg\cos T)\sin T}{(\sin 2T - 2kg)^2} \\
&, p = \sin T, T = k(x + 4k^2 t + x_1(k)), \\
&g = \partial_k T = x + 12k^2 t + y_1; \\
&k, x_1, y_1 \in R, p_k := \partial_k p = g\cos T.
\end{aligned}
$$

The unique pole $x_0(t)$ of $u(x,t)$ is determined by the formula

$$\begin{aligned}
&x_0 = -12k^2 t + y_1 + \tfrac{\delta(t)}{2k},\\
&\delta = \sin(\delta - 16k^3 t + 2k(x_1 - y_1)),\\
&\delta(t + \tfrac{2\pi}{2k}) = \delta(t).
\end{aligned}$$

The associated solution is slowly decaying at infinity, and essentially differs from well-known soliton solution.

This solution is defined as follows

$$u = -2\partial_x^2 \log W(p, \partial_k p, s),$$
$$s = \cosh Y, Y = b(x - 4b^2 t + r), \quad r \in R.$$

The related Wronskian τ-function in this formula can be computed explicitly :

$$W = 2kb\sin^2 T \sinh Y + ((b^2 - k^2)2^{-1}\sin 2T - k(k^2 + b^2)g)\cosh Y.$$

For a plot of W and its asymptotic properties, see [14, 15, 16]. It is remarkable that, for all real values of the parameters this solution has only one pole on the real axis as a function of x.

It is determined by the formula :

$$u = -2\partial_x^2 \log W(p(1), \partial_{k_1} p(1), p(2), \partial_{k_2} p(2)\,),$$
$$p(1) := \sin T_1,\, T_1 = k_1(x + x_1 + 4k_1^2 t),\; p(1)_{k_1} = g_1 \cos T_1$$
$$g_1 = x + y_1 + 12k_1^2 t;$$
$$p(2) = \sin T_2,\; T_2 = k_2(x + x_2 + 4k_2^2 t),$$
$$p(2)_{k_2} = g_2 \cos T_2, g_2 = x + y_2 + 12k_2^2 t,$$
$$\operatorname{Im} x_1 = \; y_1 = \operatorname{Im} x_2 = \operatorname{Im} y_2 = 0;\; k_{1,2} > 0$$

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