Reverse draining of a magnetic soap film

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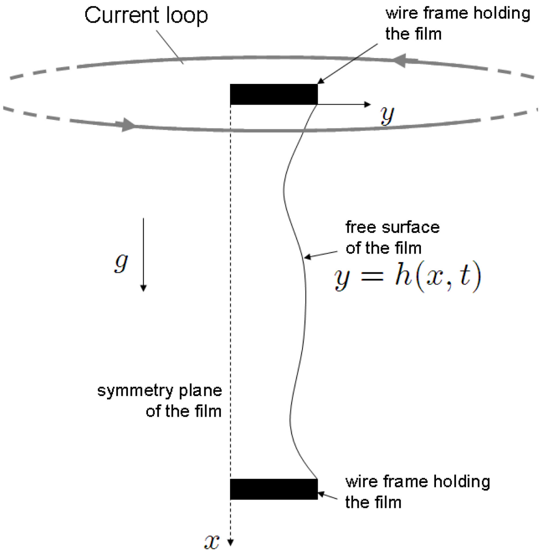
D. Moulton & J. Lega, *Reverse draining of a magnetic soap film - Analysis and simulation of thin film equation with non-uniform forcing*,
Physica D **238**, 2153-2165 (2009)

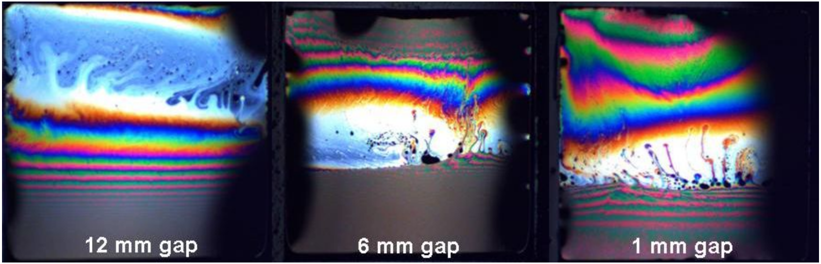
Outline

1. Introduction
   - Experiments
   - Model
   - Simulations
2. Thin film dynamics and convergence towards equilibrium solutions
3. Singular equilibrium solutions
4. Bifurcation diagrams and hysteresis
5. Summary and open questions

Introduction: Experimental setup



Introduction: Experimental observations



Experiments and photos by Derek Moulton.

In the presence of a magnetic field, black film can appear at the bottom of the draining film.

D.E. Moulton & J.A. Pelesko, *Reverse draining of a magnetic soap film*, submitted to Phys. Rev E.

Introduction: Model

- In the lubrication approximation, the dimensionless equation describing the half-thickness of the film, h, is given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left[\sigma \frac{\partial^3 h}{\partial x^3} + 1 + \lambda f(x)\right]\right) = 0.$$

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- The function $f(x) = \dfrac{-3\eta^2 x}{(1+\eta^2 x^2)^4}$ describes the effect of the magnetic field. Here, η is the ratio of the radius of the current loop to the length of the film.

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- The boundary conditions are

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- The above no-flux boundary conditions lead to conservation of the volume of the film,

$$V = \int_0^1 h(x,t)\; dx = \text{constant.}$$

- Previous results on the dynamics of thin film equations are typically obtained for zero forcing f and different boundary conditions.

Introduction: Model

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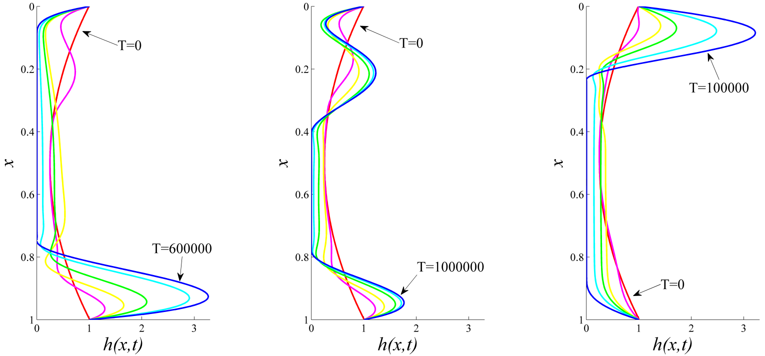
- We use a second order finite difference conservative numerical scheme.
- The initial condition is a slightly perturbed parabolic profile, parametrized by the volume of the film, which is conserved by the dynamics.

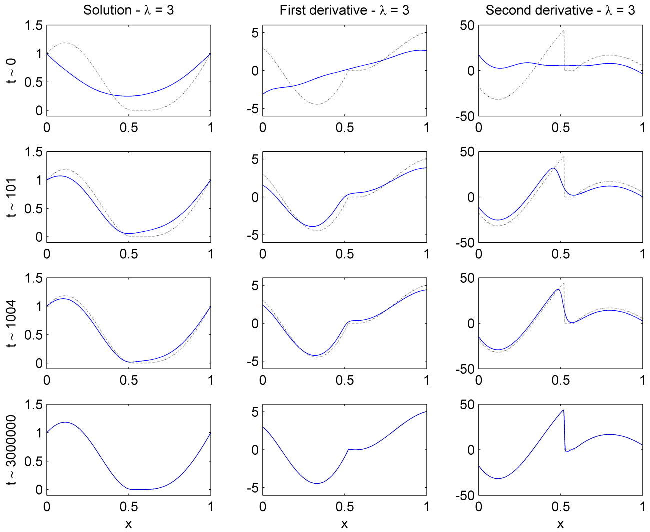
Numerical simulations



Left: relatively weak magnetic field strength ($\lambda = 1$); Middle: magnetic field of intermediate strength ($\lambda = 1.9$); Right: relatively strong magnetic field ($\lambda = 5$). In each case, $\sigma = 0.001$ and $\eta = 1$.

Our simulations show the formation of black film and suggest that the profile often evolves towards a singular equilibrium solution, which appears to be reached in an infinite amount of time.

Convergence towards singular equilibrium solution



Parameters: $\sigma = 0.004$, $\eta = 1$, and $V = 0.5$.

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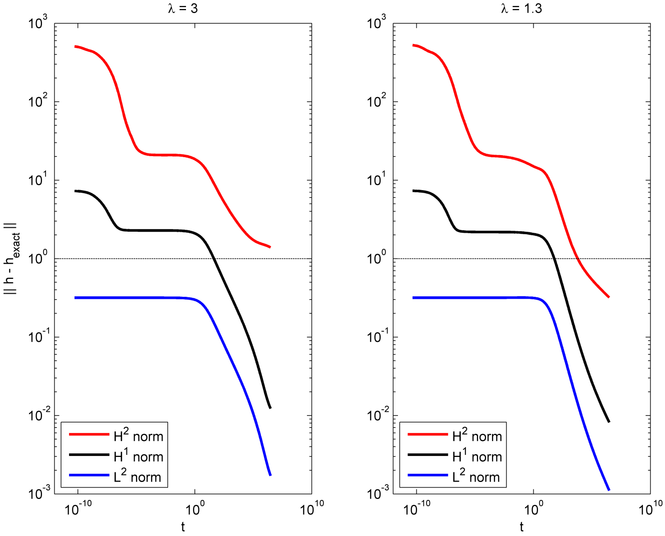

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Outline

Equilibrium solutions

- We look for equilibrium solutions that satisfy the differential equation

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- A smooth solution to $\sigma\frac{\partial^3 h}{\partial x^3} + 1 + \lambda f(x) = 0$ always exists, but it may not always satisfy $h(x) \geq 0$.
- We are therefore led to consider non-smooth solutions that satisfy, in a piecewise fashion,

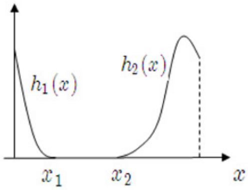
$$h(x) = 0 \qquad \text{or} \qquad \sigma\frac{\partial^3 h}{\partial x^3} + 1 + \lambda f(x) = 0.$$

Equilibrium solutions with a single zero-interval

To place this in a more general framework, we consider solutions to

$$h^3\left[\frac{\partial^3 h}{\partial x^3}-\gamma(x)\right]=0, \qquad h(0)=h(1)=1,\ \int_0^1 h(x)\,dx=V,$$

and look for piecewise solutions with one zero-interval.



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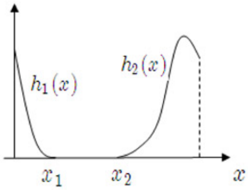

- We assume that $h(x)$ and $h'(x)$ are continuous.

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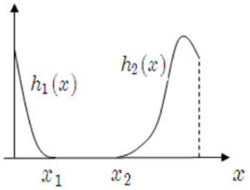

- We assume that $h(x)$ and $h'(x)$ are continuous.
- We therefore have 7 conditions for 8 unknowns (3 constants of integration for each $h_i(x)$, plus x_1 and x_2).

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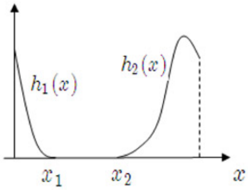
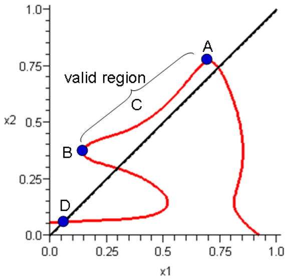
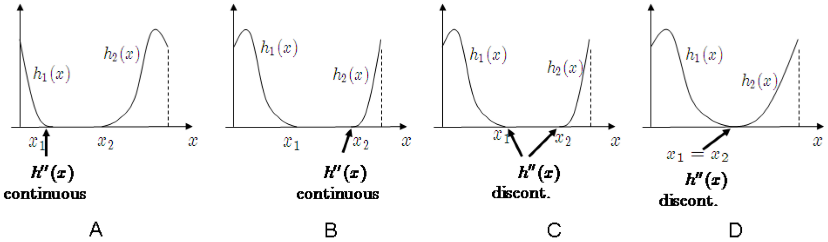


- We assume that $h(x)$ and $h'(x)$ are continuous.
- We therefore have 7 conditions for 8 unknowns (3 constants of integration for each $h_i(x)$, plus x_1 and x_2).
- This gives us a relationship between x_1 and x_2.

Equilibrium solutions with a single zero-interval



We can prove the following:



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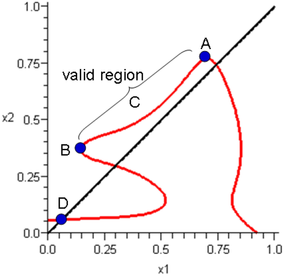
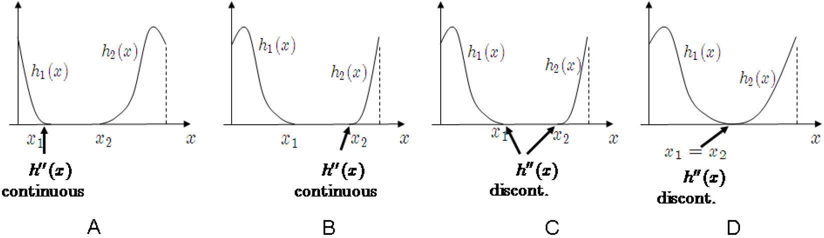


We can prove the following:

- Valid solutions are such that $\dfrac{dx_2}{dx_1} \geq 0.$



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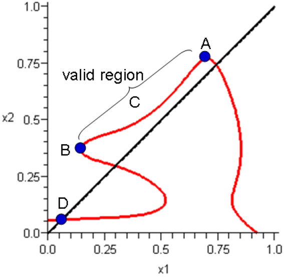
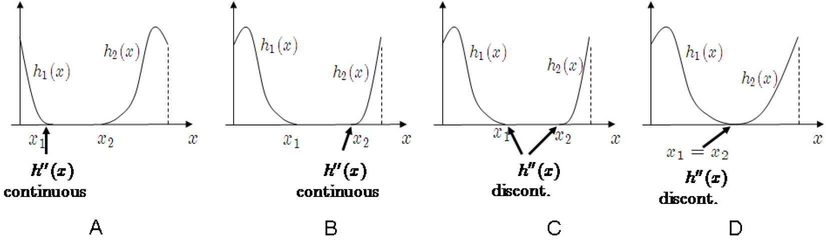


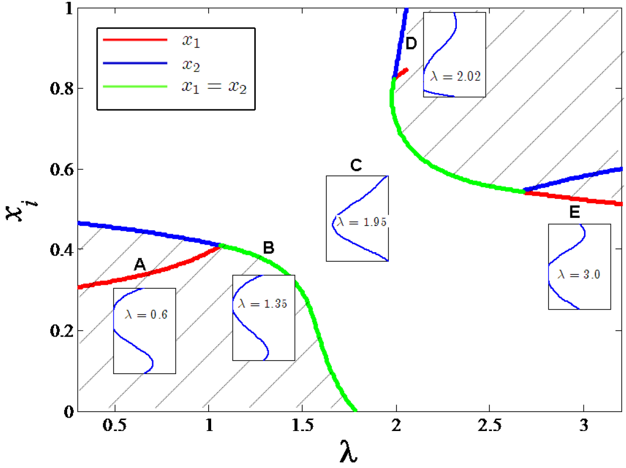
We can prove the following:

- Valid solutions are such that $\dfrac{dx_2}{dx_1} \geq 0$.
- Points A, B, and D correspond to special solutions as illustrated below.



Equilibrium solutions with a single zero-interval

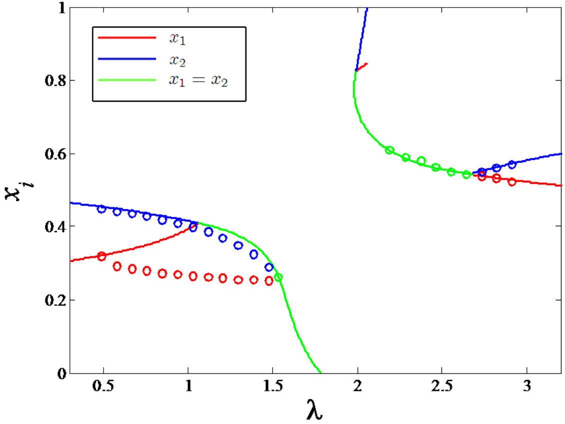
This allows us to give a complete description of how equilibrium solutions evolve with the strength of the magnetic field λ.



Parameters: $\sigma = 0.004$, $\eta = 1$, $V = 0.5$, and $\lambda \in (0.4, 3.1)$.

Simulations with increasing magnetic field

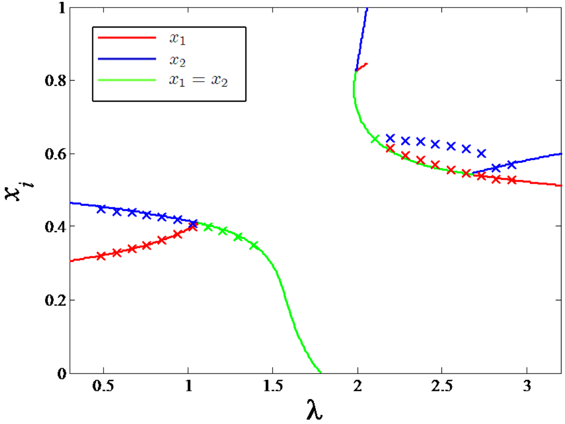
The figure below summarizes simulations in which λ is progressively increased, using the final state of the previous simulation as initial condition.



Parameters: $\sigma = 0.004$, $\eta = 1$, $V = 0.5$, and $\lambda \in (0.49, 2.9)$.

Simulations with decreasing magnetic field

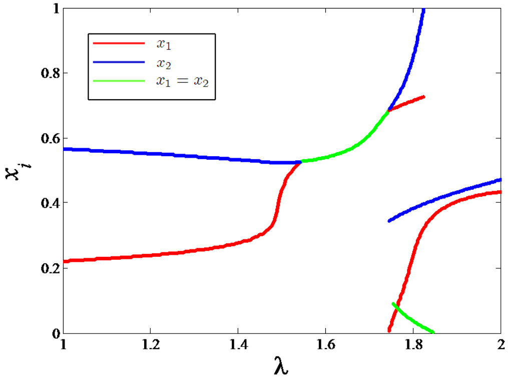
The figure below summarizes simulations in which λ is progressively decreased, using the final state of the previous simulation as initial condition.



Parameters: $\sigma = 0.004$, $\eta = 1$, $V = 0.5$, and $\lambda \in (0.49, 2.9)$.

Equilibrium solutions with a single zero-interval

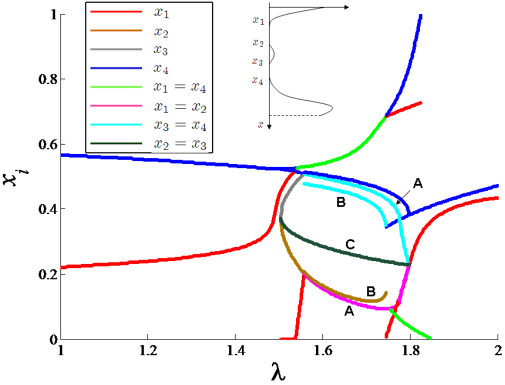
For the parameter values chosen before, only continuous solutions and solutions with one zero-interval were possible. We now decrease σ, so that solutions with two zero-intervals exist.



Parameters: $\sigma = 0.001$, $\eta = 1$, $V = 0.5$, and $\lambda \in (1, 2)$.

Equilibrium solutions with one and two zero-intervals

The bifurcation diagram becomes much more intricate. For the magnetic forcing f, we can show that solutions with more than two zero-intervals are not possible.



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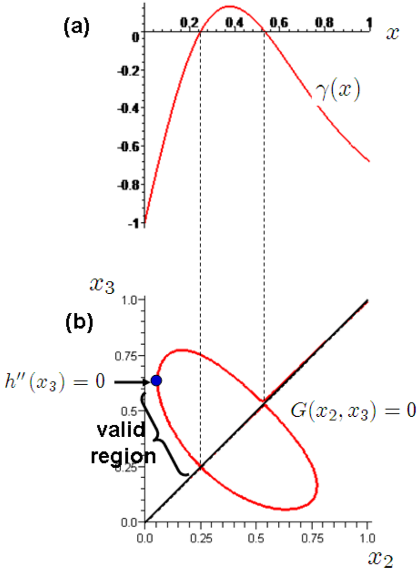
External forcing and number of zero-intervals

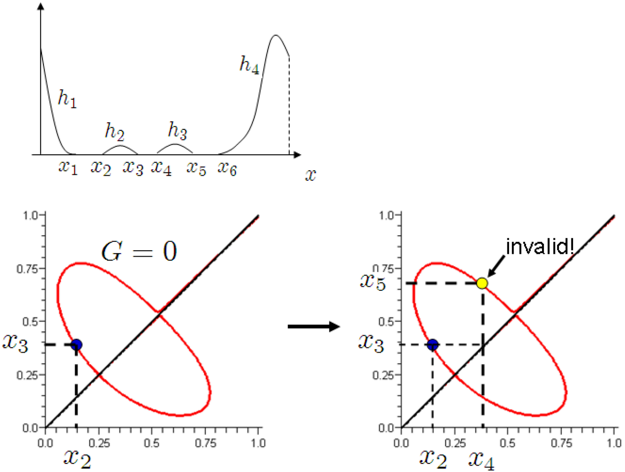


- We can prove the following:

  For solutions with at least two zero-intervals, solutions corresponding to the middle section are valid if $\gamma'(x^*) > 0$ and invalid if $\gamma'(x^*) < 0$.

- This fact can be used to obtain an upper bound on the number of possible zero-intervals in a given solution.

External forcing and number of zero-intervals (continued)



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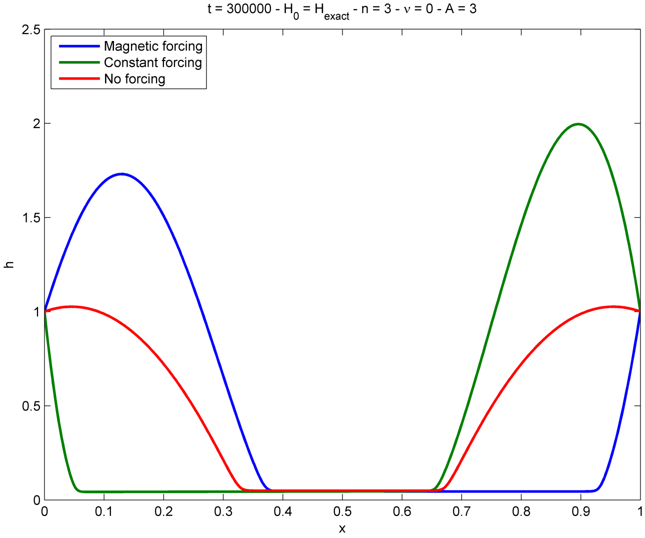
Regularization



Solution profiles for different forcings and in the presence of disjoining pressure.

Regularization

The previous slide shows solutions of the thin film equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} Q(x, t) = 0$$

with

$$Q(x, t) = h^3 \left(\frac{1}{3} \left[\sigma \frac{\partial^3 h}{\partial x^3} + 1 - \lambda f(x) + \frac{\partial \Pi}{\partial x} \right] \right).$$

The disjoining pressure $\Pi(x, t)$ is given by

$$\Pi(x, t) = A \left(\left(\frac{h^*}{h} \right)^4 - \left(\frac{h^*}{h} \right)^3 \right),$$

with $A = 3$ and $h^* = 0.05$.

Introduction: Model

- There is ample literature on the dynamics of the thin-film equation
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(f(h)\, \frac{\partial^3 h}{\partial x^3} \right) = 0.$$

- In particular, with periodic boundary conditions and strictly positive initial data, the above equation with $f(h) = |h|^n$, $n > 0$, is shown to have, for $t > t_c$, a strong positive solution that tends towards its mean as $t \to \infty$.

A.L. Bertozzi & M. Pugh, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Commun. Pure Appl. Math. **49** 85-123 (1996)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(f(h)\,\frac{\partial^3 h}{\partial x^3}\right) = 0$$

- With boundary conditions of the form

$$\frac{\partial h}{\partial x} = \frac{\partial^3 h}{\partial x^3} = 0 \qquad \text{at } x = 0 \text{ and } x = 1,$$

the above equation with $f(h) \sim |h|^n$, $2 \leq n \leq 4$, has a weak solution such that if the initial data is positive, then the solution remains positive almost everywhere for arbitrary times.

F. Bernis & A. Friedman, *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations **83** 179-206 (1990)

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