"Phase transitions" for solitons

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OUTLINE

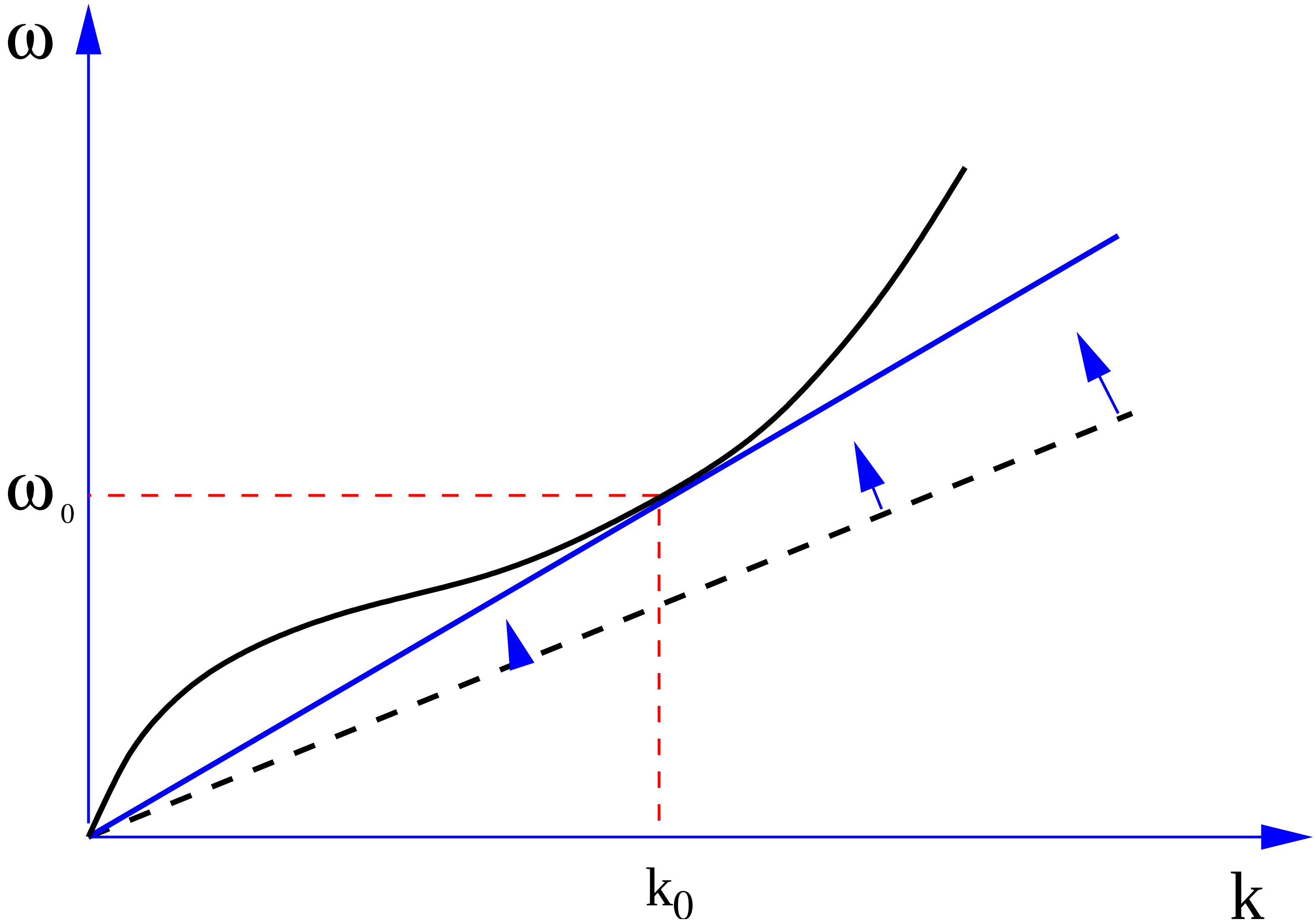
- Introduction. History of the problem
- Basic equations
- Hamiltonian expansion and matrix elements
- Solitary wave solutions: bifurcations, stability and collapse
- Conclusion

Introduction. History of the problem

- Longuet-Higgins [1989] provided numerical evidence of 1D capillary-gravity (CG) solitary waves in deep water. These solitons exist if their velocity is less the minimal phase velocity V_{cr} of linear waves.
- If the soliton velocity $V > V_{cr}$ then soliton will be in resonance with linear waves and due to Cherenkov radiation of small amplitude waves will lose its energy and finally disappear. At $V = V_{cr}$ solitons undergo bifurcation. What kind of bifurcations depends on physical model.

Introduction. History of the problem

$\omega = \sqrt{gk + \sigma k^3}$.



Introduction. History of the problem

- Iooss & Kirchgässner [1990] proved the existence of 1D CG solitary waves in deep water within the full water-wave equations.
- As it was established by Akylas [1993] and Longuet-Higgins [1993] as $V \to V_{cr}$ the CG solitary waves are transformed into the envelope solitons of the NLS equation.
- In the water-wave case, the bifurcation is always supercritical. Near the supercritical bifurcation soliton behaves universally always: the soliton amplitude vanishes as $(V_{cr} - V)^{1/2}$ and its width grows like $(V_{cr} - V)^{-1/2}$ (Zakharov & Kuznetsov [1998]).

Introduction. History of the problem

- For interfacial waves (propagating along the density jump), the bifurcation can be subcritical: solitons at the bifurcation point undergo a jump in amplitude (Dias & Iooss[1996]).
- The transition from the supercritical bifurcation to the subcritical one occurs due to the nonlinearity changes from focusing to defocusing.
- Such a transition for interfacial solitons happens at the density ratio ρ equal to $\rho_{cr} = (21 - 8\sqrt{5})/11 \approx 0.283$ (Dias & Iooss[1996]).

Introduction. History of the problem

- For optical system such situation can happen for anomalous dispersion if the 4-wave coupling coefficient becomes small, or even negative, for instance, due to the striction effect (interaction with acoustic phonons) (Kuznetsov [1999]).
- More reach situation takes place for bifurcations in the system of interactive basic and second harmonics in nonlinear optical media (Grimshaw, Kuznetsov, Shapiro [2001]).

Supercritical bifurcation

Consider a nonlinear wave medium which can be described by

$$H = \int \omega_k |a_k|^2 d\mathbf{k} + H_{int},$$

The equations of motion in the normal amplitudes a_k are written in the standard form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i\frac{\delta H_{int}}{\delta a_k^*},$$

so that in the absence of an interaction $a_k(t) = a_k(0)e^{-i\omega_k t}$. In x-space introduce the psi-function,

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{d/2}} \int a_k(t) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}.$$

Supercritical bifurcation

If $\psi(\mathbf{x}, t)$ is a periodic function of the coordinates, then its spectrum $a_k(t)$ consists of a collection of δ-functions. For localized distributions $\psi(\mathbf{x}, t) \to 0$ as $|\mathbf{x}| \to \infty$, the Fourier amplitude $a_k(t)$, being a localized function of $\mathbf{k}$, does not contain δ-function singularities.

For soliton propagating with constant velocity $\mathbf{V}$,
$\psi(\mathbf{x}, t) = \psi(\mathbf{x} - \mathbf{V}t)$, $a_k(t) = c_k e^{-i\mathbf{k}\cdot\mathbf{V}t}$, where

$$(\omega_k - \mathbf{k} \cdot \mathbf{V})c_k = -\frac{\delta H_{int}}{\delta c_k^*} \equiv f_k.$$

In order $\omega_k - \mathbf{k} \cdot \mathbf{V} \geq 0$ for all $\mathbf{k}$ the soliton velocity has to be less than the minimum phase velocity: $|\mathbf{V}| < \min(\omega_k / k)$.

This is the main selection rule for solitons.

Supercritical bifurcation

Let us assume the equation $\omega_k = \mathbf{k} \cdot \mathbf{V}$ has a solution at $\mathbf{k} = \mathbf{k}_0$. Then,

$$c_k = A\delta(\mathbf{k} - \mathbf{k}_0) + \frac{f_k}{\omega_k - \mathbf{k} \cdot \mathbf{V}}, \;\; f_{k_0} = 0.$$

This equation can be solved by iterations, taking $A\delta(\mathbf{k} - \mathbf{k}_0)$ as the zeroth term. As the result, the solution will take the form

$$c_k = \sum_n A_n \delta(\mathbf{k} - n\mathbf{k}_0),$$

i.e. the solution is periodic in x-space.

Thus, to have soliton solution $\omega_k - \mathbf{k} \cdot \mathbf{V}$ must be sign-definite, which is equivalent to the absence of Cherenkov radiation.

Supercritical bifurcation

Thus, for existence of solitons the equation

$$c_k = \frac{f_k}{\omega_k - \mathbf{k}\cdot\mathbf{V}}.$$

has no zeros in the dominator. As $V \to V_{cr}$ ($V_{cr} = \min(\omega/k)$), c_k possesses a sharp peak at the touching point:

$$c_k = \left[\frac{1}{2}\omega_{\alpha\beta}\kappa_\alpha\kappa_\beta + k_0(V_{cr} - V)\right]^{-1} f_k.$$

Here $\omega_{\alpha\beta} = \partial^2\omega/\partial k_\alpha \partial k_\beta$ is a symmetric, positive-definite, tensor of the second derivatives, evaluated at $\mathbf{k} = \mathbf{k}_0$, and $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{k}_0$.

Supercritical bifurcation

As $V \to V_{cr}$ the width $\Delta k \propto \sqrt{V_{cr} - V}$, the distribution approaches a monochromatic wave.
If the soliton amplitude vanishes as $V \to V_{cr}$ then the solution $\psi(\mathbf{x})$ can be sought as a sum of harmonics:

$$\psi(\mathbf{x}') = \sum_{h=-\infty}^{\infty} \psi_n(\mathbf{X}) e^{in\mathbf{k}_0 \cdot \mathbf{x}'}, \quad \mathbf{x}' = \mathbf{x} - \mathbf{V}t.$$

Here $\lambda = \sqrt{1 - V/V_{cr}} \ll 1$, $\mathbf{X} = \lambda \mathbf{x}'$ is the “slow” coordinate , $\psi_n(\mathbf{X})$ the amplitude of the envelope of n-th harmonic. Hence, we arrive at the stationary NLS:

$$-k_0 V_{cr} \lambda^2 \psi_1 + \frac{1}{2} \omega_{\alpha\beta} \frac{\partial^2 \psi_1}{\partial X_\alpha \partial X_\beta} + B |\psi_1|^2 \psi_1 = 0,$$

Supercritical bifurcation

Here $B = -(2\pi)^d \tilde{T}_{k_0 k_0 k_0 k_0}$ is the 4-wave coupling coefficient. After performing a rotation to the principal axes of $\omega_{\alpha\beta}$ and carrying out the corresponding extensions, the equation takes the standard form

$$-\lambda^2 \psi + \Delta \psi - \mu |\psi|^2 \psi = 0,$$

where $\mu = \text{sign}(\tilde{T}\omega_{\alpha\alpha})$.

Hence:

i) solitons are possible only if $\mu < 0$ (focusing nonlinearity),

ii) the soliton amplitude $\propto \sqrt{V_{cr} - V}$,

iii) the soliton size $\propto (V_{cr} - V)^{-1/2}$.

At $V = V_{cr}$ the approximation becomes precise.

Supercritical bifurcation

The number of particles (or intensity) on a soliton solution as a function of λ has the form

$$N = \int |\psi|^2 d\mathbf{x} \propto \lambda^{2-d},$$

where d is the space dimension. Hence, according to the Vakhitov-Kolokolov criterion $\partial N/\partial\lambda^2 > 0$ we get stable solitons only in 1D.

This is probably the simplest method for explaining the well-known empirical fact that solitons, as a rule, exist only in one-dimensional systems. For multidimensional systems stable solitons are rare and can appear as a result of only topological constraints or a mechanism that removes Cherenkov singularities.

Bifurcations for interfacial waves

- Two ideal fluids with different densities $\rho_{1,2}$ in the presence of gravity (g) acting down the vertical z-axis and capillarity with interfacial tension σ.
- The light fluid, labeled 1, occupies the region $\infty > z > \eta(x,t)$, and respectively the heavy one occupies the region $-\infty < z < \eta(x,t)$.
- Flows of both fluids are assumed to be potential and 2D:

$$\mathbf{v}_{1,2} = \nabla \phi_{1,2},$$

Potentials $\phi_{1,2}$ satisfy the Laplace equation:

$$\Delta \phi_{1,2} = 0$$

Boundary conditions

- On the interface $z = \eta(x,t)$ there are two (kinematic and dynamic) boundary conditions:

$$\frac{\partial \eta}{\partial t} = \left(-\eta_x \frac{\partial \phi_{1,2}}{\partial x} + \frac{\partial \phi_{1,2}}{\partial z} \right)_{z=\eta} \equiv U,$$

$$\rho_2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2} (\nabla \phi_2)^2 + g\eta \right) - \rho_1 \left(\frac{\partial \phi_1}{\partial t} + \frac{1}{2} (\nabla \phi_1)^2 + g\eta \right)$$
$$= \sigma \frac{\partial}{\partial x} \left(\frac{\eta_x}{\sqrt{\eta_x^2 + 1}} \right).$$

Hamiltonian formulation

- The equations can be represented in the Hamiltonian form with $\Psi = (\phi_1 - \rho\phi_2)|_{z=\eta}$ and η as canonical variables:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi}, \quad \frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$

  This is the generalization of Zakharov [1968] canonical form to interfacial waves (Kontorovich & Sinitsyn [1974], Kuznetsov & Spector [1976]).

- The Hamiltonian coincides with the total energy:

$$H = \frac{1}{2}\int \left[U\Psi + (1-\rho)g\eta^2 + 2\sigma\left(\sqrt{1+\eta_x^2} - 1\right)\right] dx.$$

  Here $\rho = \rho_1/\rho_2$ is the density ratio.

Hamiltonian formulation

- In the normal variables a_k, the equations become

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}.$$

- The Hamiltonian can be expanded in infinite series with respect to powers of a_k and a_k^*. This expansion, $H = H_0 + H_{int}$, starts from $H_0 = \int \omega_k |a_k|^2 dk$, where

$$\omega_k = \left(\frac{|k|}{1+\rho} [g(1-\rho) + \sigma k^2] \right)^{1/2}.$$

is the linear dispersion. H_{int} is responsible for nonlinear wave interaction.

Stationary NLSE

- The solitary wave velocity V must be less the minimum phase velocity of linear waves,

$$V_{cr} = \min \frac{\omega_k}{k} \equiv \frac{\omega_0}{2k_0},$$

(attained at the point $k = k_0$) and $\omega_0 \equiv \omega(k_0)$.

- $\omega'' = \partial^2\omega/\partial k^2 = \omega_0/(2k_0^2) > 0$.
- At the leading order in λ, one obtains the stationary NLSE:

$$-\lambda^2\omega_0\psi + \frac{\omega_0}{4k_0^2}\frac{\partial^2\psi}{\partial x^2} - \mu|\psi|^2\psi = 0,$$

Stationary NLSE

- The interaction Hamiltonian H_{int} is restricted to

$$\overline{H}^{(4)} = \frac{\mu}{2} \int |\psi|^4 \, dx.$$

- For interfacial waves

$$\mu = \frac{k_0^3}{1+\rho} \left(A_{cr}^2 - A^2 \right)$$

- $A = (1-\rho)/(1+\rho)$ is the Atwood number, and $A_{cr}^2 = 5/16$.

Stationary NLSE

- For $\rho < \rho_{cr}$, $\mu < 0$, the nonlinearity is of focusing type and solitons undergo supercritical bifurcation: their amplitude vanishes proportionally to $\sqrt{V_{cr} - V}$ with increasing width like $(V_{cr} - V)^{-1/2}$.
- For $\rho > \rho_{cr}$ nonlinearity in the SNLS becomes defocusing ($\mu > 0$) and therefore, to get soliton solutions, one needs to keep next order terms beyond the classical NLSE.
- The next order terms come from the expansion of 4- and 6-wave interactions.

$$\overline{H}^{(4)} = \frac{1}{2} \int \left[\mu|\psi|^4 + 2i\beta(\psi_x^* \psi - \psi_x \psi^*)|\psi|^2 - \gamma|\psi|^2 \widehat{k|}\, \psi|^2 \right] \, dx$$

Defocusing nonlinearity

- $$\gamma = \frac{k_0^2}{1+\rho}, \quad \beta = \frac{3k_0^2}{16(1+\rho)}.$$
- $\widehat{k} = -\partial_x \widehat{H}$ with $\widehat{H}$ the Hilbert transform.
- The second term is analogous to the Lifshitz term in phase transition theory.
- At $\rho = 0$ the nonlocal term coincides with that obtained first by Dysthe for gravity waves.
- Due to positiveness of γ the nonlocal interaction is of focusing type that can provide wave localization.

Extended NLSE

- Calculations show that the contribution from six-wave interaction is also of the focusing type:

$$\overline{H}^{(6)} = -C \int |\psi|^6 \, dx.$$

- The coupling coefficient C is found with account of all renormalizations due to three-, four- and five-wave interactions:

$$C = \frac{Mk_0^6}{\omega_0}, \quad M = \frac{289(21 + 8\sqrt{5})}{49152} \approx 0.228654.$$

Solitons of extended NLS equation and their bifurcations

- Soliton envelope is found from the variational problem, $\delta(\overline{H} + \lambda^2\omega_0 N) = 0,$ where the (averaged) Hamiltonian $\overline{H}$ and the number of waves N are given by the expressions:

$$\overline{H} = \frac{1}{2}\int \omega''|\psi_x|^2 dx + \overline{H}^{(4)} + \overline{H}^{(6)}, \qquad N = \int |\psi|^2\, dx.$$

- The variational problem leads to

$$-\lambda^2\omega_0\psi + \frac{\omega_0}{4k_0^2}\psi_{xx} - \mu|\psi|^2\psi$$

$$+4i\beta|\psi|^2\psi_x + \gamma\psi\hat{k}|\psi|^2 + 3C|\psi|^4\psi = 0.$$

Solitons of extended NLS equation and their bifurcations

- In terms of amplitude and phase, $\psi = re^{i\varphi}$, the Hamiltonian in dimensionless variables has the form

$$H = \int \left[r_x^2 + \frac{\mu}{2} r^4 - \frac{\gamma}{2} r^2 \widehat{k} r^2 - \frac{1}{3} r^6 + r^2 \left(\varphi_x + \beta r^2 \right)^2 \right] dx,$$

where the renormalized constant $\tilde{C} = C + \beta^2$ (= 1/3).

- Hence H takes its minimum value when

$$\varphi_x + \beta r^2 = 0.$$

Integrating this equation gives an x-dependence for the phase, called chirp in nonlinear optics.

Solitons of extended NLS equation and their bifurcations

- The soliton amplitude $r(x)$ is defined from

$$-\lambda^2 r + r_{xx} - \mu r^3 + r^5 + \gamma r\hat{k}(r^2) = 0,$$

where $\mu = Sign(\rho - \rho_{cr})$.

- Below the critical density ratio ($\mu = -1$) all nonlinear terms are of the focusing type. As the result, solitons exist in the whole range of λ and we arrive at a supercritical bifurcation. The soliton amplitude

$$r = \frac{\sqrt{2}\lambda}{\cosh(\lambda x)}$$

vanishes $\sim \lambda$ as $\lambda \to 0$.

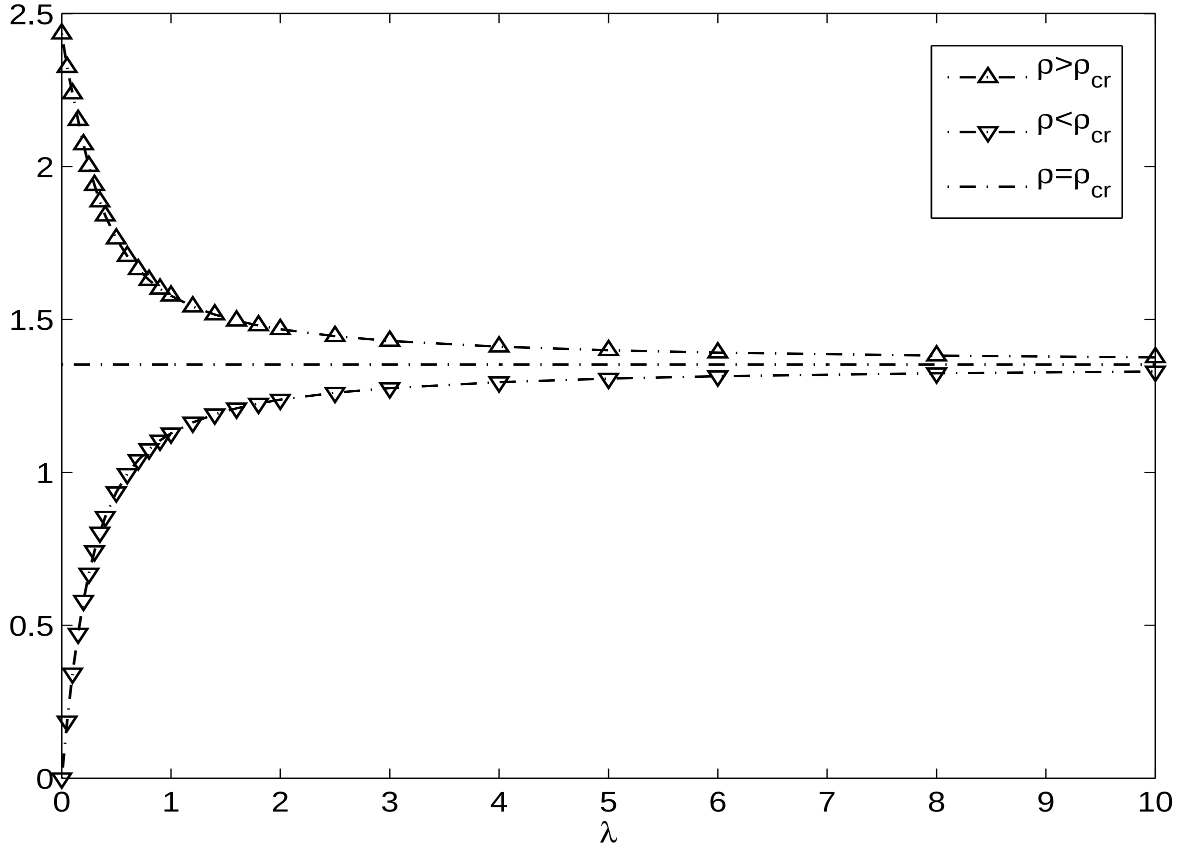
Solitons of extended NLS equation and their bifurcations

- Above the critical density ratio solitons undergo a subcritical bifurcation.
- Explicit soliton solution with finite amplitude can be found at $\lambda = 0$:

$$r^2 = \frac{2}{x^2 + 4/3}.$$

This is a soliton with algebraic decay at infinity.

Numerical solitary wave solutions



Dependencies of N on λ for $\mu = 0, \pm 1$

Stability criterion

- The dependence $N(\lambda)$ gives possibility to make some predictions about soliton stability based on the Vakhitov-Kolokolov criterion.
- If the derivative $\partial N_s/\partial\lambda^2$ on the soliton family is negative then solitons are unstable and they are stable in opposite situation.
- For the lower soliton branch at $\lambda \to 0$ this criterion gives stability and instability for solitons with $\mu > 0$.
- However, the Vakhitov-Kolokolov criterion, derived for the classical NLSE, can not be applied to our system strictly speaking.

Estimate of stability for the upper branch

- Use Lyapunov theorem
- Consider the dependence of the Hamiltonian $\overline{H}$ evaluated on the class of the functions with fixed N.
- Under the scaling scaling transformation

$$r = \frac{1}{a^{1/2}} r_s \left(\frac{x}{a} \right),$$

$\overline{H}$ becomes a function of the scaling parameter a:

$$\overline{H}\,(a) = \left(\frac{1}{a} - \frac{1}{2a^2} \right) \frac{\mu}{2} \int r_s^4 dx; \qquad \overline{H}_s \equiv \overline{H}\,(a=1) = \frac{\mu}{4} \int r_s^4 dx,$$

Estimate of stability for the upper branch

- At $\mu > 0$ this function is unbounded from below as $a \to 0$ and has a maximum $\overline{H}_s > 0$ corresponding to the soliton solution.
- Another transformation of the gauge type, $\psi = \psi_s e^{i\chi}$, demonstrates that this soliton represents a saddle point of the Hamiltonian and therefore the upper soliton branch is unstable (at least, with respect to finite perturbations).

Estimate of stability for the lower branch

- For the lower soliton branch these both transformations give a minimum.
- One can show that the lower branch solitons indeed achieves a minimum of the Hamiltonian $\overline{H}$ for fixed N.
- By means of the inequalities of Sobolev type, one finds

$$\overline{H} \geq -\frac{N^3}{4\sqrt{3}}\left[1 - \left(\frac{\gamma}{2}\frac{N}{N_2} + \frac{N^2}{3N_1^2}\right)\right],$$

if $N \leq N_3 = \sqrt{\frac{9}{16}\gamma^2\frac{N_1^4}{N_2^2} + 3N_1^2} - \frac{3}{4}\gamma\frac{N_1^2}{N_2}$.

- In accordance with the Lyapunov theorem solitons from the lower branch with $N \leq N_3$ are stable not only with respect to small perturbations but also against finite ones.

Collapse of solitons

- At $\mu = 0$ in this case this model can be considered as the critical NLS (critical in the same sense as 2D cubic NLS).
- Another reason for collapse existence in the GNLS is connected with unboundedness of the Hamiltonian from below (only for defocusing cubic nonlinearity!).

$$H(a) = \left(\frac{1}{a} - \frac{1}{2a^2}\right)\frac{\mu}{2}\int |\psi_s|^4\,dx$$

- For $\beta = 0$ the virial theorem is valid:

$$R_{tt} = 8\left(H - \frac{\mu}{4}\int |\psi|^4 dx\right),\ R = \int x^2|\psi|^2 dx,$$

At $\mu > 0$ collapse takes place for $H \leq H_s$.

Collapse of solitons

- Numerics at $\mu > 0$ show self-similar behavior for collapsing soliton. Near the collapse point the term $\sim \mu$ can be neglected. In this case GNLS admits self-similar solutions,

$$r(x,t) = (t_0 - t)^{-1/4} f\left(\frac{x}{(t_0 - t)^{1/2}}\right).$$

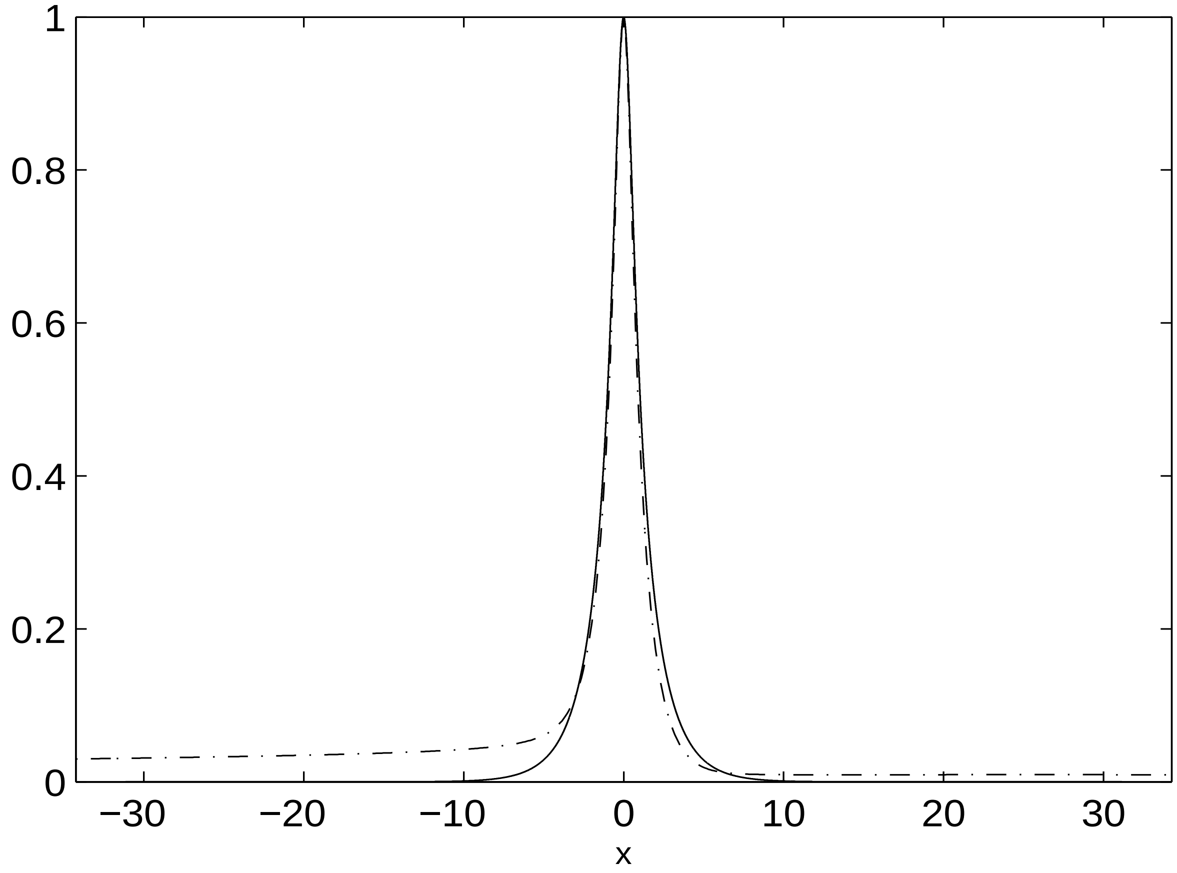
- To verify the asymptotic behavior, we normalized ψ by $\max|\psi| \equiv M$:

$$\psi(x,t) = M\overline{\psi}(\xi,\tau), \ \xi = M^2(x - x_{max}), \ \ \tau = \log M.$$

New more convinient variables do not require the determination of the collapsing time t_0.

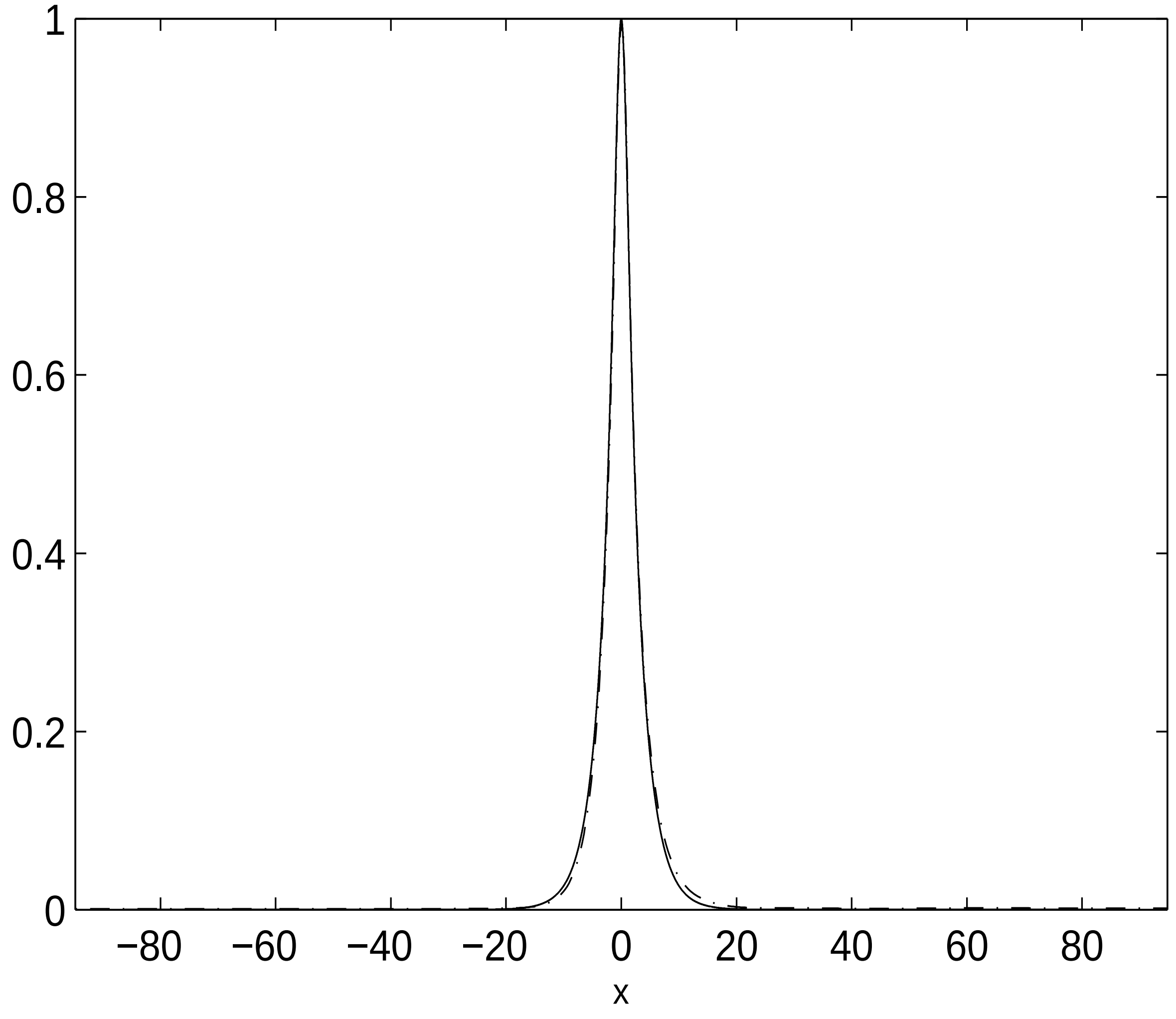
Collapse of solitons: numerical results

IW: Initial (solid line) and final (dashed line) at $t = 1.18$ dependences $|\overline{\psi}|$ on ξ. The ratio between final and initial soliton amplitudes in the physical variables is about 11.



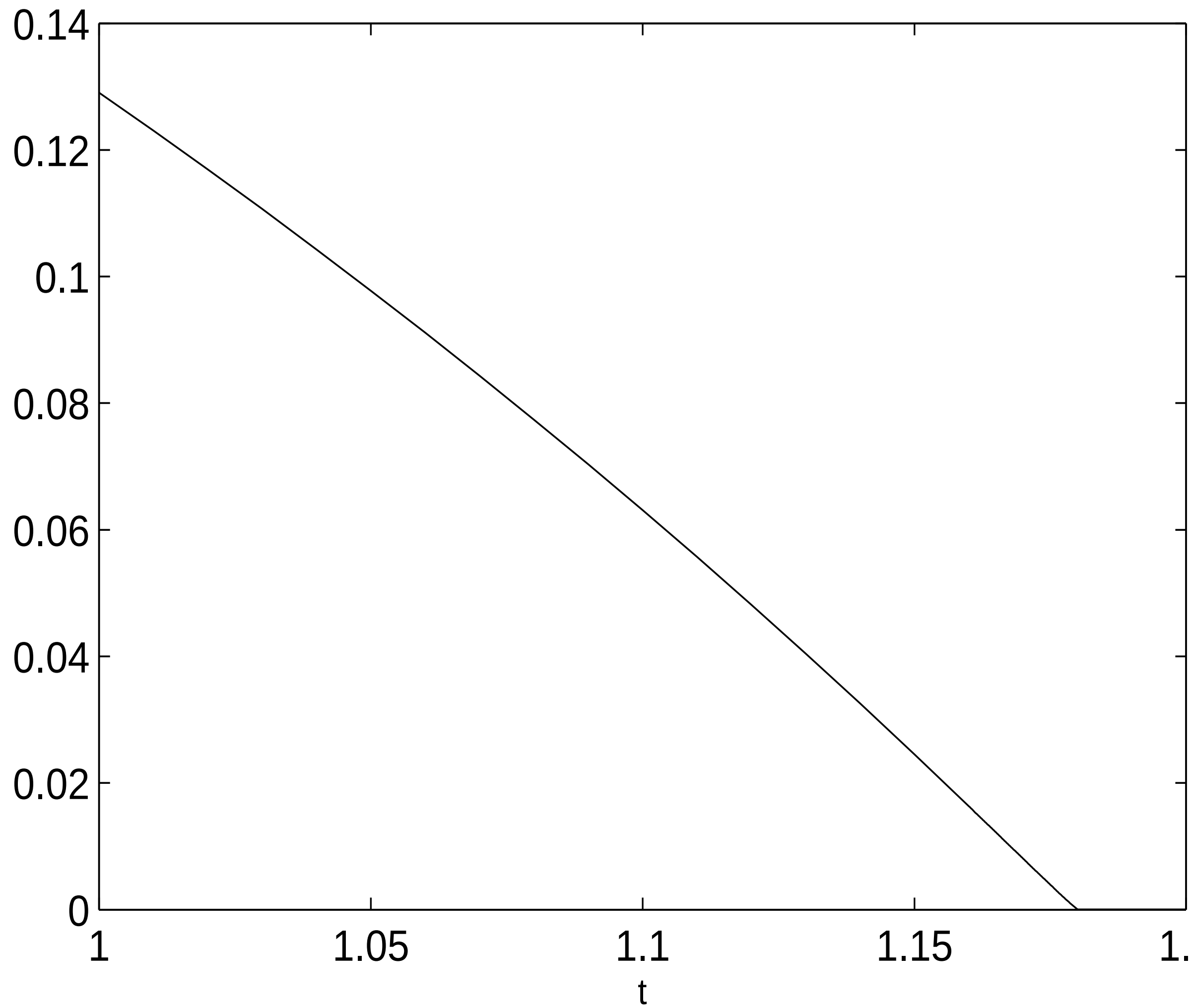
Collapse of solitons: numerical results

WW ($k_0 h \approx 1.363$): Initial (solid line) and final (dashed line) at $t = 2.7192$ dependences $|\overline{\psi}|$ on ξ.



Collapse of solitons: numerical results

Dependence of $1/\max|\psi|^4$ on time. Interfacial waves.



Conclusions

- Analysis of behavior of solitons near the critical density ratio ρ_{cr}.
- Above ρ_{cr} solitons undergo a subcritical bifurcation with the amplitude jump $\sim \sqrt{\rho - \rho_{cr}}$.
- Therefore our theory based on the Hamiltonian expansion works when this jump is small, i.e. near ρ_{cr}.
- Derivation of GNLS equation based on the Hamiltonian average over the fast oscillations with the carrying frequency corresponding to the critical soliton velocity $V = V_{cr}$.
- The generalized NLS equation contains three kinds of nonlinear terms including one nonlocal contribution.

Conclusions

- We analyzed the stability of solitons. In particular, we have found a region of wave intensity, $N \leq N_3$, where solitons stable in accordance with the Lyapunov theorem. Their stability is based on the boundedness of H from below.
- For solitons above ρ_{cr}: possible instability, at least with respect to finite perturbations.
- Nonlinear stage of this instability leads to wave collapse. The form of the central part of the collapsing pulse in self-similar variables approaches the soliton shape. Asymptotically collapse tends to the critical one for NLS systems.
- There exists some similarity between the generalized NLS equation derived for IW and those describing the behavior of short optical pulses in fibers from the femtosecond range of pulse durations.

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