

On Dissipation Function of Ocean Waves due to Whitecapping.

V.E. Zakharov, A.O. Korotkevich, and A.O. Prokofiev

Department of Mathematics, The University of Arizona, Tucson;

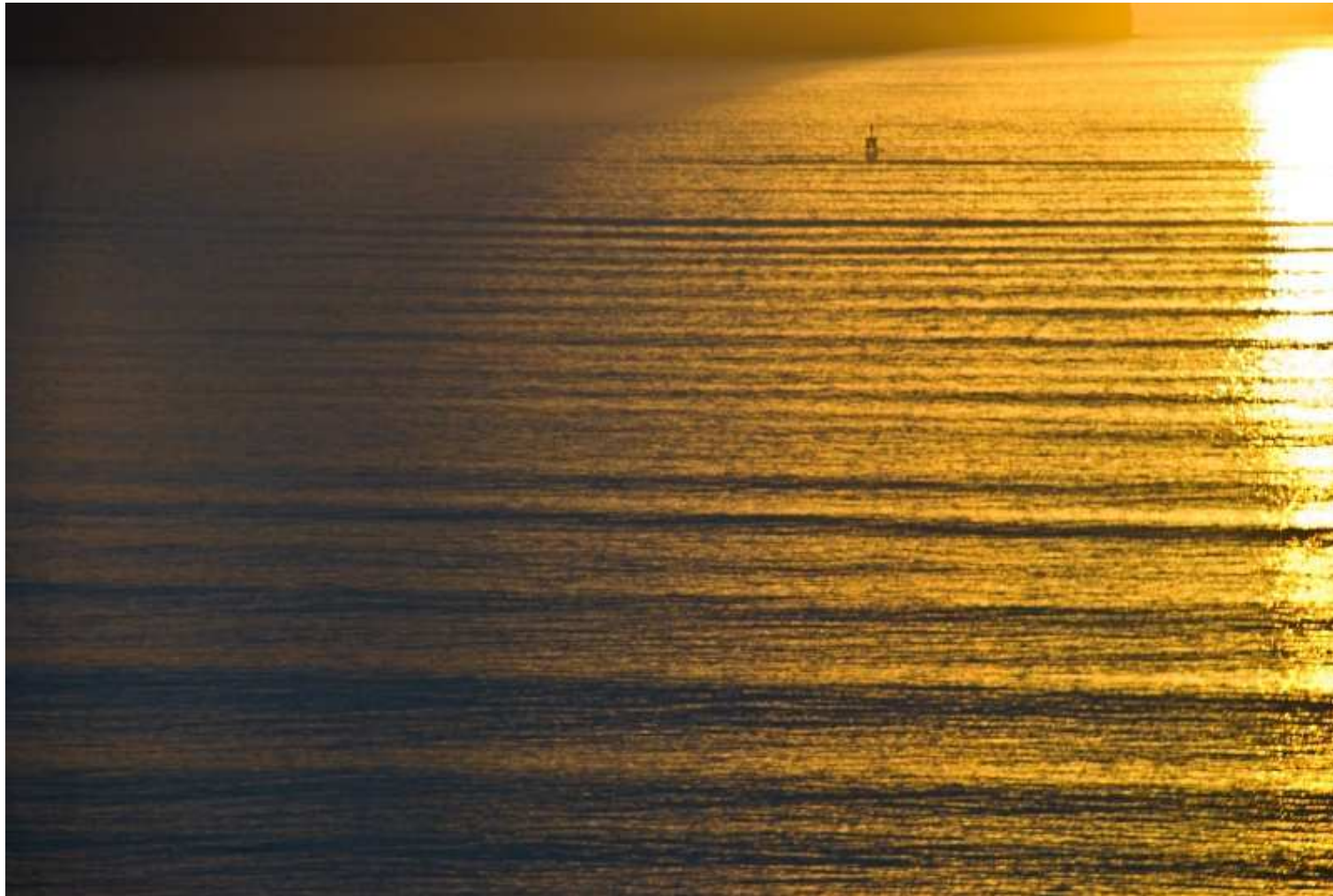
Department of Mathematics and Statistics, The University of New Mexico, Albuquerque.

L.D. Landau Institute for Theoretical Physics RAS, Moscow;

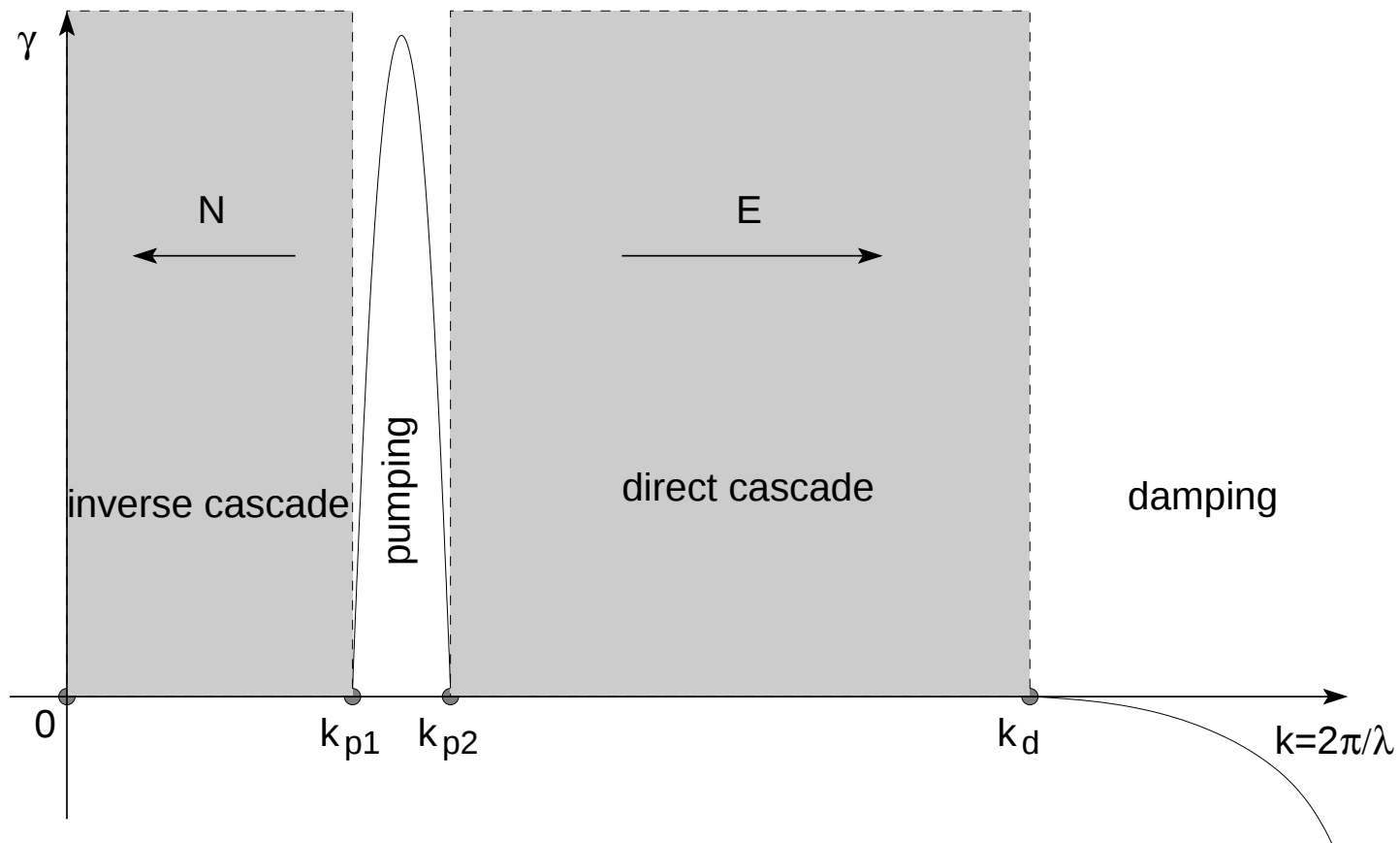
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On Dissipation Function of Ocean Waves due to White Capping.

Waves forecasting.

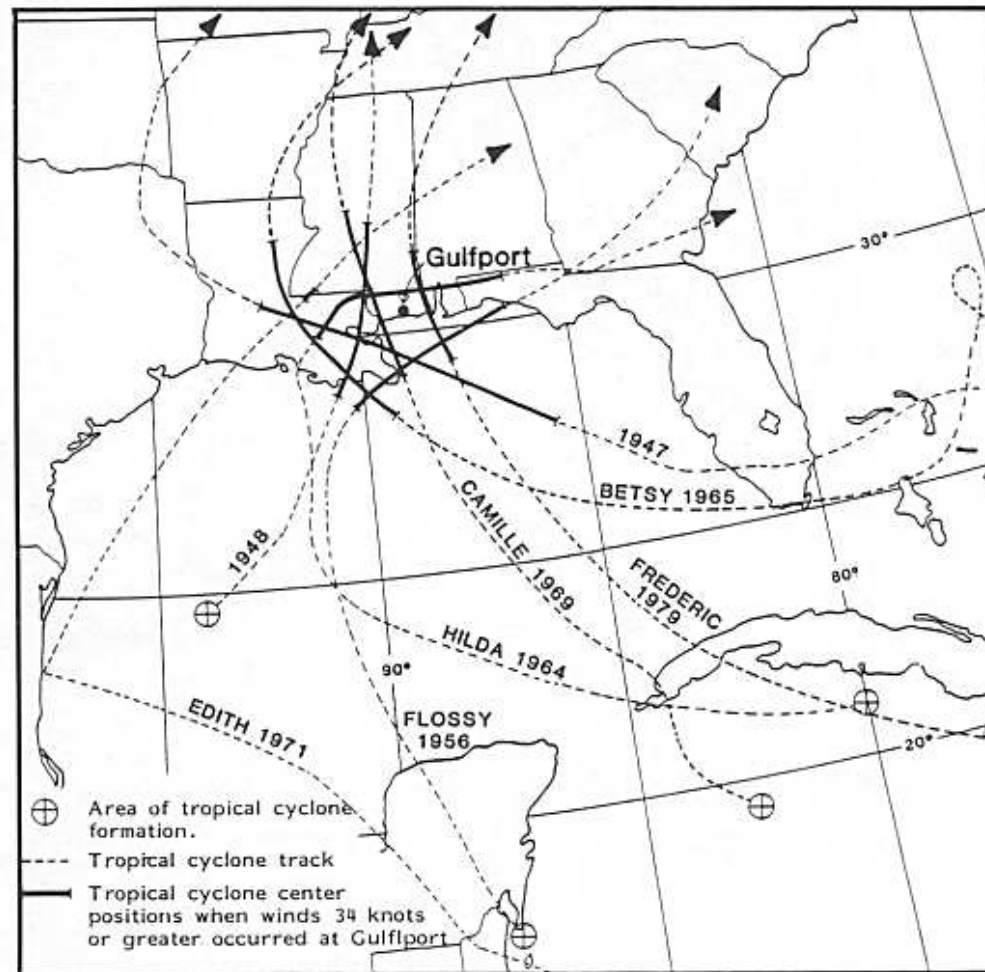


Scheme of scales



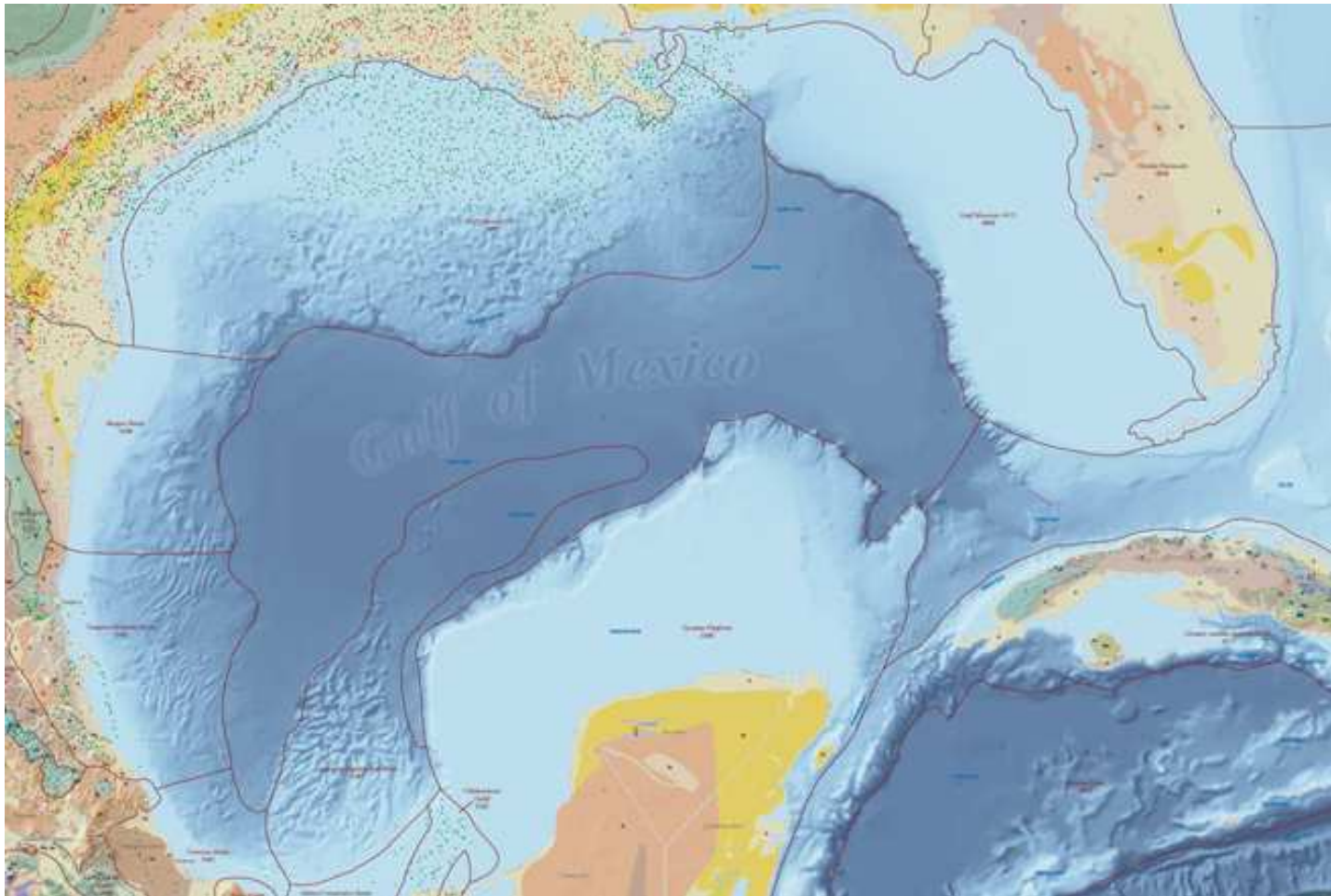
On Dissipation Function of Ocean Waves due to White Capping.

Why it is important?



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Purpose of wave forecasting



Kinetic equation

The pair correlation function for excitations N_k obeys the kinetic equation (Nordheim, 1929; Hasselmann, 1962; Zakharov, 1966)

$$\frac{\partial N_k}{\partial t} = st(N, N, N) + f_p(k) - f_d(k), \quad (1)$$

Here

$$\begin{aligned} st(N, N, N) = & 4\pi \int \left| T_{\vec{k}, \vec{k}_1, \vec{k}_2, \vec{k}_3} \right|^2 \times \\ & \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - \\ & - N_k N_{k_1} N_{k_3}) \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3. \end{aligned} \quad (2)$$

The kinetic equation and its modifications are the base for all wave forecasting models.

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White capping.



Dissipation function.

Dissipative part of kinetic equation

$$\frac{\partial N_{\vec{k}}}{\partial t} = \dots - \gamma_{\vec{k},\mu}^{kin} \omega_k N_{\vec{k}}. \quad (3)$$

If $N_{\vec{k}}$ is almost monochromatic (swell) we can find dependence of γ^{kin} on average steepness μ :

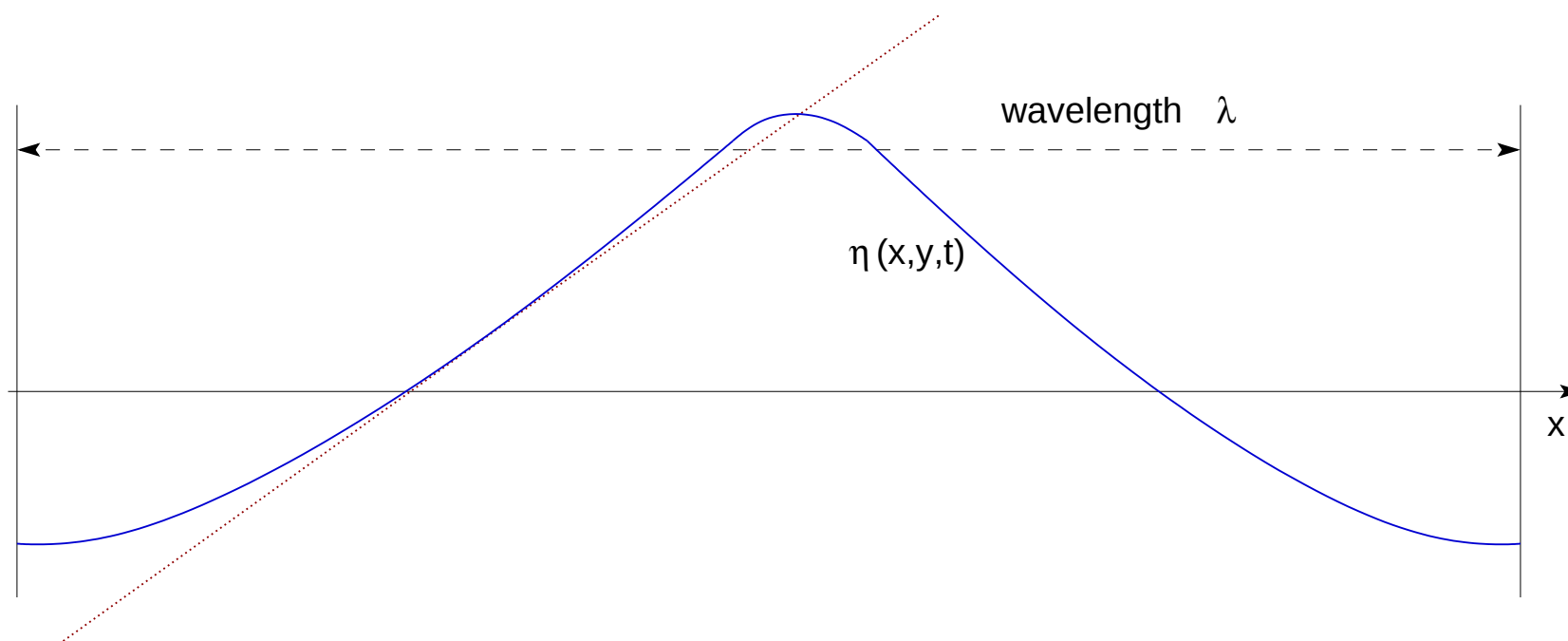
$$\gamma^{kin}(\mu) = -\frac{\dot{N}}{\omega_p N}, \quad (4)$$

Here

$$N = \int n_{\vec{k}} d^2 k.$$

Problem formulation

Let us consider a potential flow of an ideal fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t)$, $\vec{r} = (x, y)$; $\vec{v} = \nabla\phi$ and surface elevation $\eta(\vec{r}, t)$.



Steepness of the surface $\mu = \sqrt{\langle |\nabla\eta(\vec{r}, t)|^2 \rangle}$ — average slope of the surface.

Energy of the system

Fluid flow is incompressible $(\nabla \vec{v}) = \Delta \phi = 0$. The total energy of the system can be presented in the following form

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2} \int d^2r \int_{-\infty}^{\eta} (\nabla \phi)^2 dz, \quad (5)$$

Potential energy due to gravity:

$$U = \frac{1}{2} g \int \eta^2 d^2r, \quad (6)$$

here g is the gravity acceleration.

Hamiltonian expansion

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad (7)$$

where $\psi = \phi(\vec{r}, \eta(\vec{r}, t), t)$ is a velocity potential on the surface of the fluid. In order to calculate the value of ψ we have to solve the Laplace equation in the domain with varying surface η . One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here $\Delta = \nabla^2$ and $\hat{k} = \sqrt{-\Delta}$)

$$\begin{aligned} H = & \frac{1}{2} \int \left(g\eta^2 + \psi \hat{k} \psi \right) d^2r + \\ & + \frac{1}{2} \int \eta \left[|\nabla \psi|^2 - (\hat{k} \psi)^2 \right] d^2r + \\ & + \frac{1}{2} \int \eta (\hat{k} \psi) \left[\hat{k} (\eta (\hat{k} \psi)) + \eta \Delta \psi \right] d^2r. \end{aligned} \quad (8)$$

Dynamical equations

In this case dynamical equations acquire the following form

$$\begin{aligned}\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ &\quad + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - D_{\vec{r}}, \\ \dot{\psi} &= -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\ &\quad - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - D_{\vec{r}} + F_{\vec{r}}.\end{aligned}\tag{9}$$

Here $D_{\vec{r}}$ is some artificial damping term used to provide dissipation at small scales; $F_{\vec{r}}$ is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

$$\psi_{\vec{k}} = \frac{1}{2\pi} \int \psi_{\vec{r}} e^{i\vec{k}\vec{r}} d^2r, \quad \eta_{\vec{k}} = \frac{1}{2\pi} \int \eta_{\vec{r}} e^{i\vec{k}\vec{r}} d^2r.$$

Numerical scheme parameters

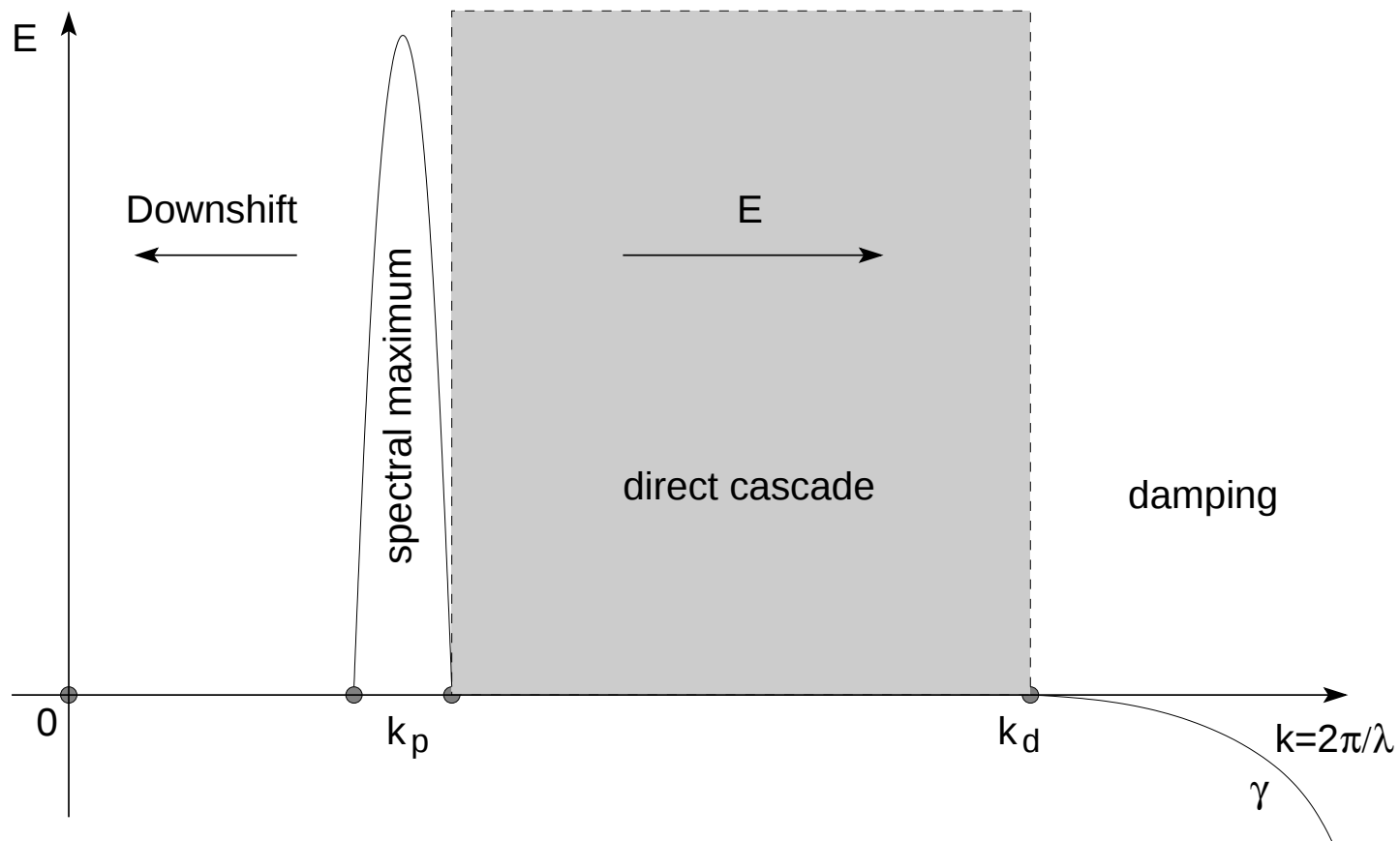
Let us add pseudo-viscous damping in dynamical equations

$$\begin{aligned}\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\ &\quad + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - F^{-1}[\gamma_k\eta_{\vec{k}}], \\ \dot{\psi} &= -g\eta - \frac{1}{2} \left[(\nabla\psi)^2 - (\hat{k}\psi)^2 \right] - \\ &\quad - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - F^{-1}[\gamma_k\psi_{\vec{k}}].\end{aligned}\tag{10}$$

$$\begin{aligned}\gamma_k &= \gamma_0(k - k_d)^2, k > k_d, \gamma_0 = 2.86 \times 10^{-3}; \\ \gamma_k &= 0, k \leq k_d.\end{aligned}\tag{11}$$

Gravity acceleration $g = 1$. Simulation region $L_x = L_y = 2\pi$ with double periodic boundary conditions. Rectangular numerical grid $N_x = 512$, $N_y = 4096$. Pseudo-viscous dissipation starts at $k_d = 1024$. Time step $\Delta t = 4.23 \times 10^{-4}$

Scheme of scales

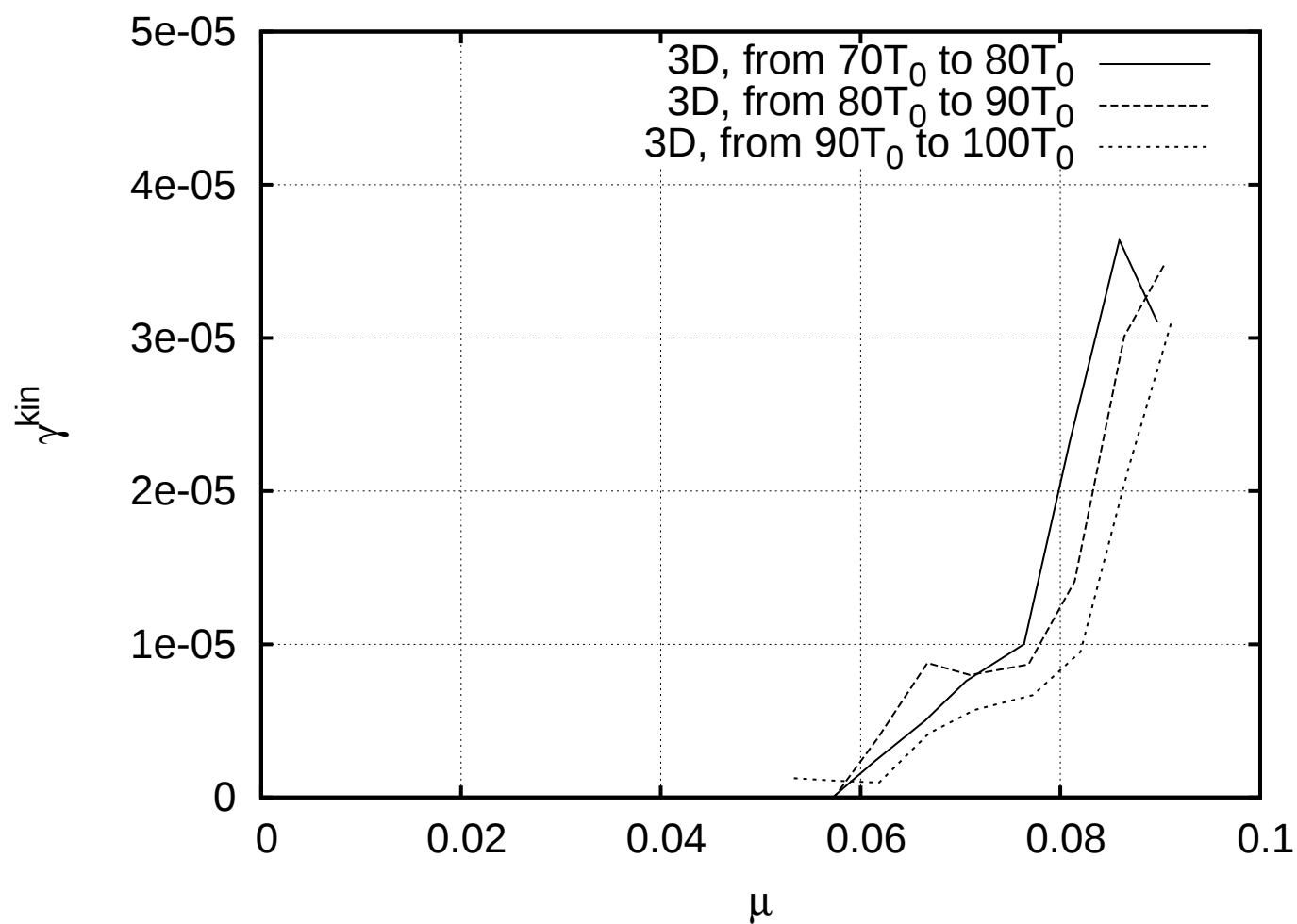


Initial conditions.

Gauss-shaped spectrum, centered at $\vec{k} = (0; 100)$ with width $D = 30$.

$$\left\{ \begin{array}{l} |a_{\vec{k}}| = A_i \exp \left(-\frac{1}{2} \frac{|\vec{k} - \vec{k}_0|^2}{D_i^2} \right), \quad |\vec{k} - \vec{k}_0| \leq 2D_i, \\ |a_{\vec{k}}| = 10^{-12}, \quad |\vec{k} - \vec{k}_0| > 2D_i. \end{array} \right. \quad (12)$$

Dissipation function. The first experiment.



Proposed energy transfer mechanism. The first experiment.

Mechanism:

- High steepness $\rightarrow$ wide spectrum $\rightarrow$ energy quickly delivered to the dissipation region and dissipated completely.

Problem:

- Weakly nonlinear model $\rightarrow$ we cannot model wavebreaking or whitecapping in details which are strongly nonlinear phenomena.

Model of energy transfer mechanism. The first experiment.

Remedy for a problem:

- We don't need to know wavebreaking **details**, because we need to know how much energy was dissipated, instead of how in details it was dissipated.
- Multiple harmonics generation describes spectrum widening during early stage of wavebreaking and whitecapping. This nonresonant mechanism is taken into account in our dynamic equations.
- Due to the universal mechanism of the spectrum widening we can check our results in the fully nonlinear 2D-model, result should be the same.

Fully nonlinear 2D experiment.

Suppose that incompressible fluid covers the domain

$$-\infty < y < \eta(x, t).$$

The flow is potential and incompressible, hence $v = \nabla\phi$, $\nabla v = 0$, $\Delta\phi = 0$.

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2} \int dx \int_{-\infty}^{\eta} (\nabla\phi)^2 dy, \quad (13)$$

Potential energy due to gravity:

$$U = \frac{1}{2}g \int \eta^2 dx \quad (14)$$

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Hamiltonian equations.

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad (15)$$

One can perform the conformal transformation to map the domain that is filled with fluid,

$$-\infty < x < +\infty, \quad -\infty < y < \eta(x, t), \quad z = x + iy,$$

in z -plane to the lower half-plane

$$-\infty < u < +\infty, \quad -\infty < v < 0, \quad w = u + iv,$$

in w -plane.

Hilbert transformation.

Now, the shape of surface $\eta(x, t)$ is presented by parametric equations

$$y = y(u, t), \quad x = x(u, t).$$

where $x(u, t)$ and $y(u, t)$ are related through Hilbert transformation

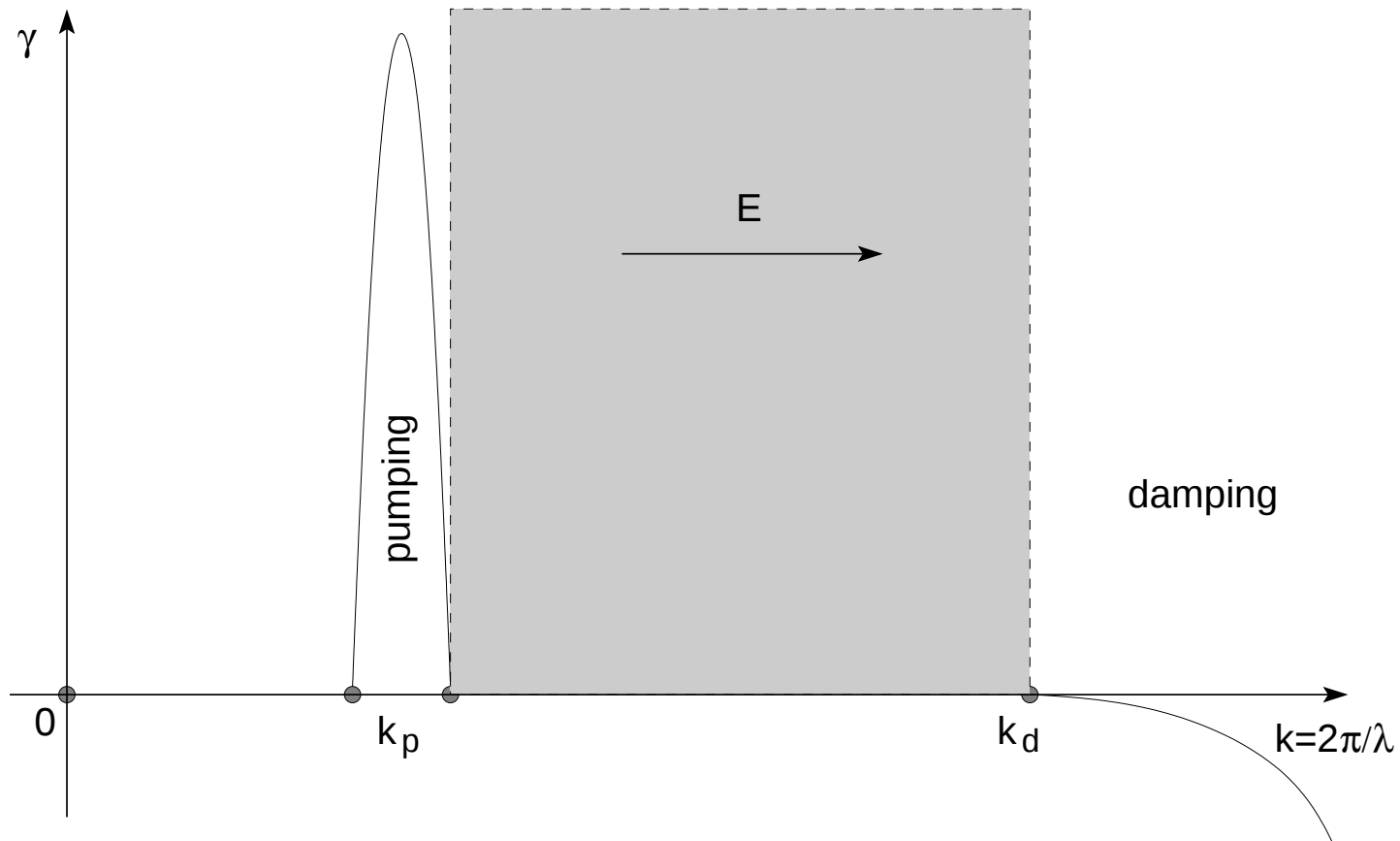
$$y(u, t) = \hat{H}(x(u, t) - u), \quad x(u, t) = u - \hat{H}(y(u, t)).$$

$$\hat{H}(f(u)) = \frac{1}{\pi} v.p. \int_{-\infty}^{+\infty} \frac{f(u') du'}{u' - u}$$

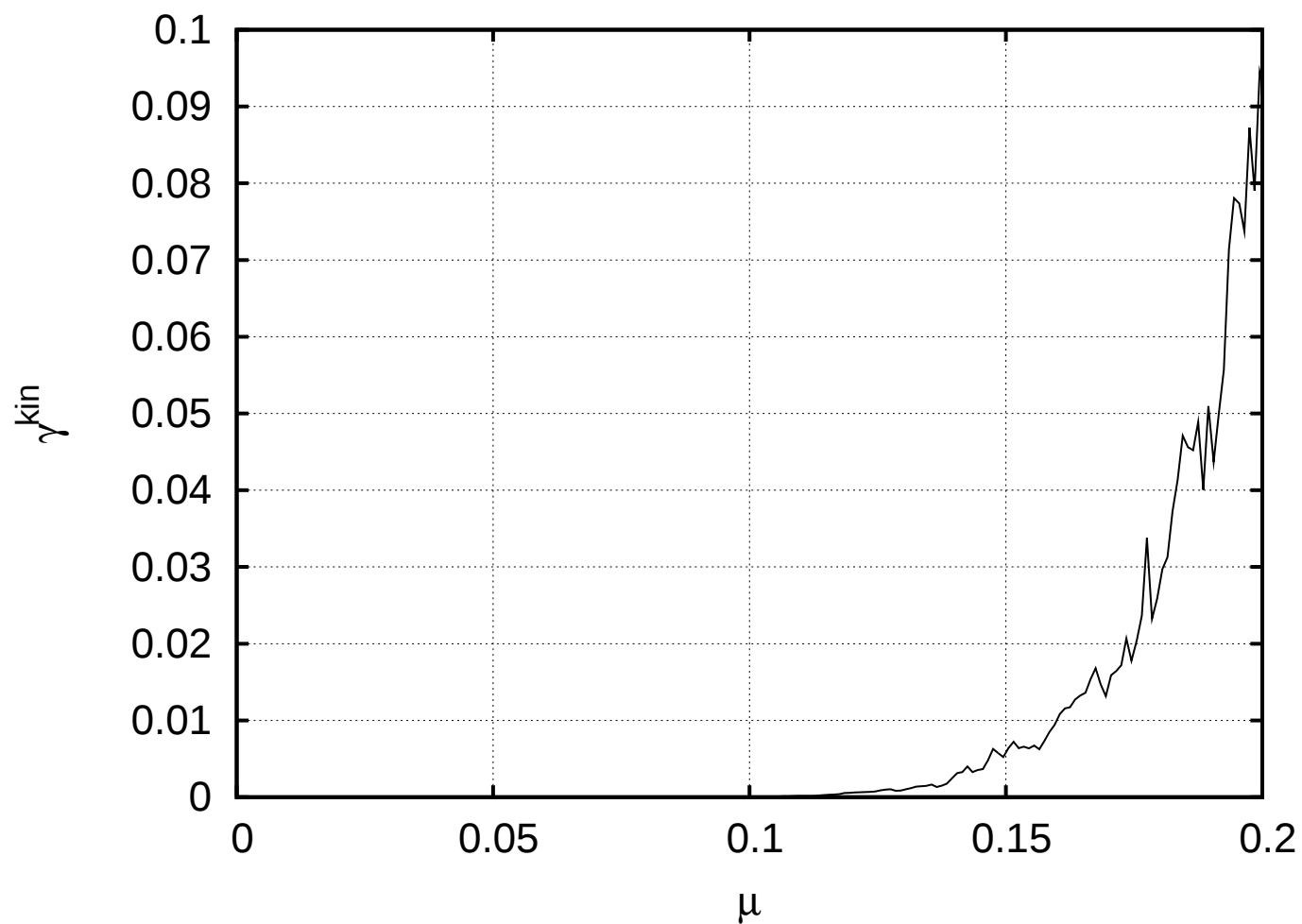
Kinetic energy term in new variables:

$$\left. \frac{\partial \Phi}{\partial v} \right|_{v=0} = -\frac{\partial}{\partial u} \hat{H} \Phi.$$

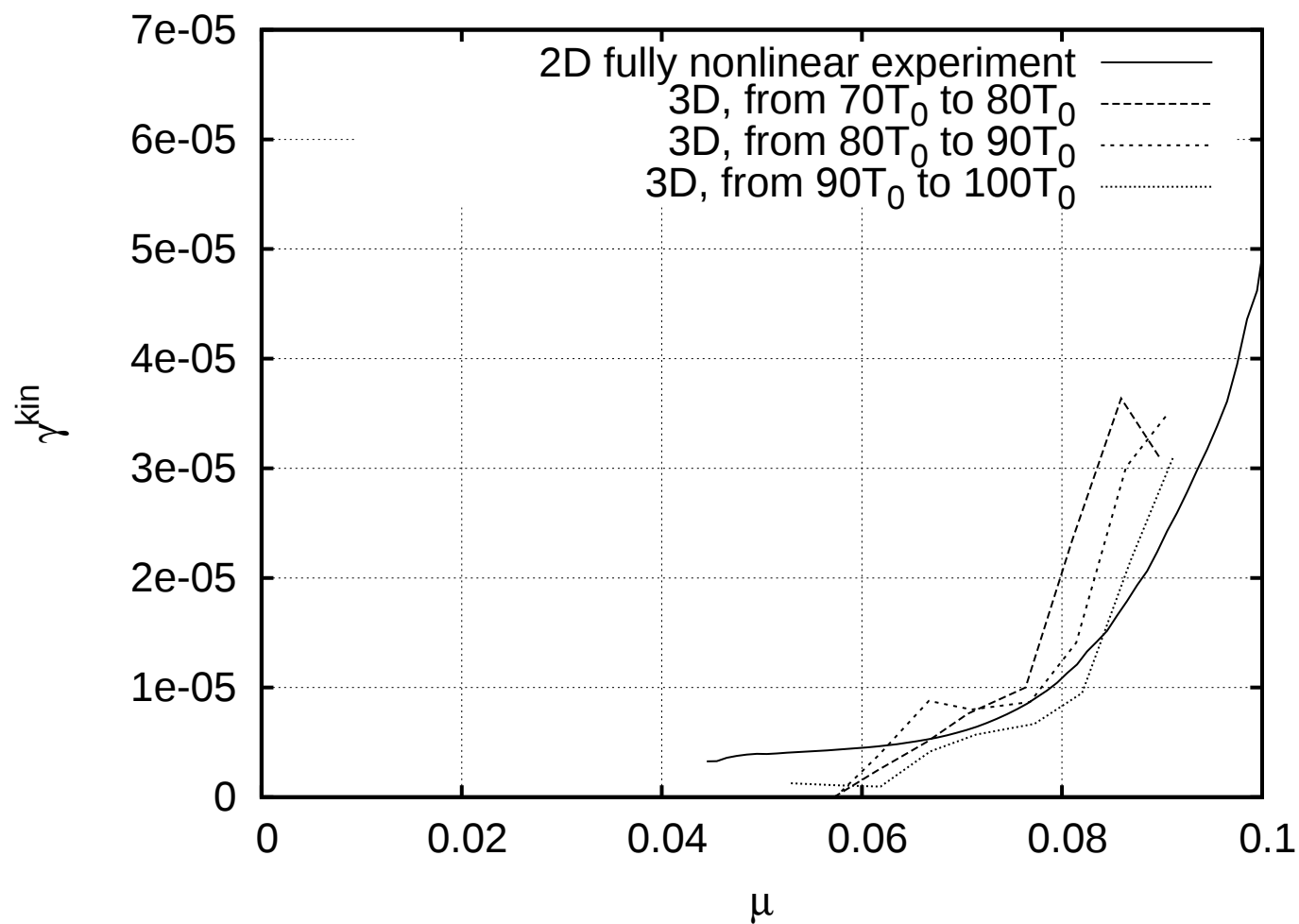
Scheme of scales



Dissipation function. The second experiment.



Dissipation function. Both experiments.



Waves forecasting models.

$$\gamma_{\vec{k}} = C_{ds} \tilde{\omega} \frac{k}{\tilde{k}} \left((1 - \delta) + \delta \frac{k}{\tilde{k}} \right) \left(\frac{\tilde{S}}{\tilde{S}_{pm}} \right)^p \quad (16)$$

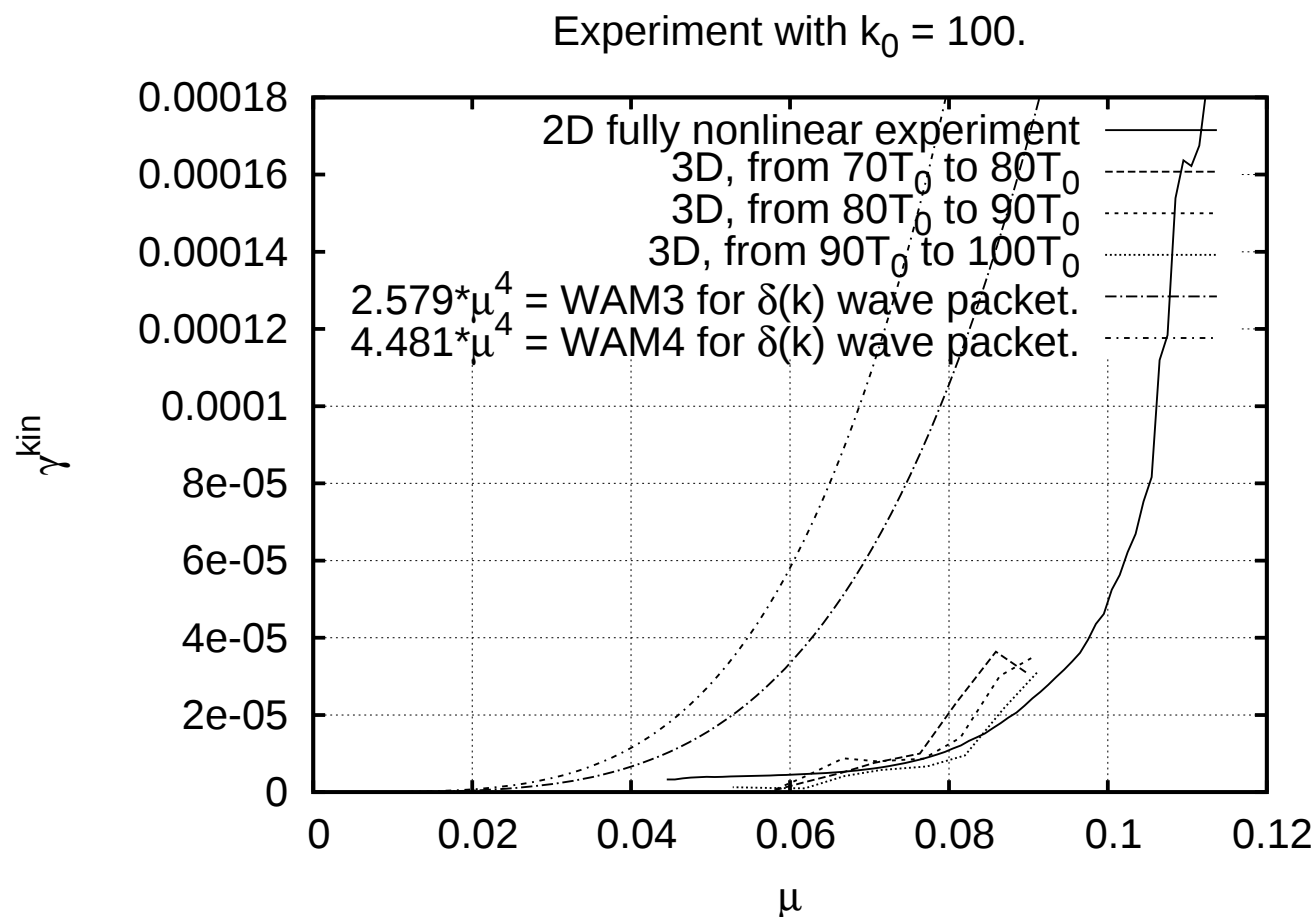
where k and ω are the wave number and frequency, tilde denotes mean value; C_{ds} , δ and p are tunable coefficients; $S = \tilde{k} \sqrt{H}$ is the overall steepness; $\tilde{S}_{PM} = (3.02 \times 10^{-3})^{1/2}$ is the value of $\tilde{S}$ for the Pierson-Moscowitz spectrum (note that the characteristic steepness is $\mu = \sqrt{2}S$). The values of the tunable coefficients for the *WAM3* case are:

$$C_{ds} = 2.35 \times 10^{-5}, \quad \delta = 0, \quad p = 4 \quad (17)$$

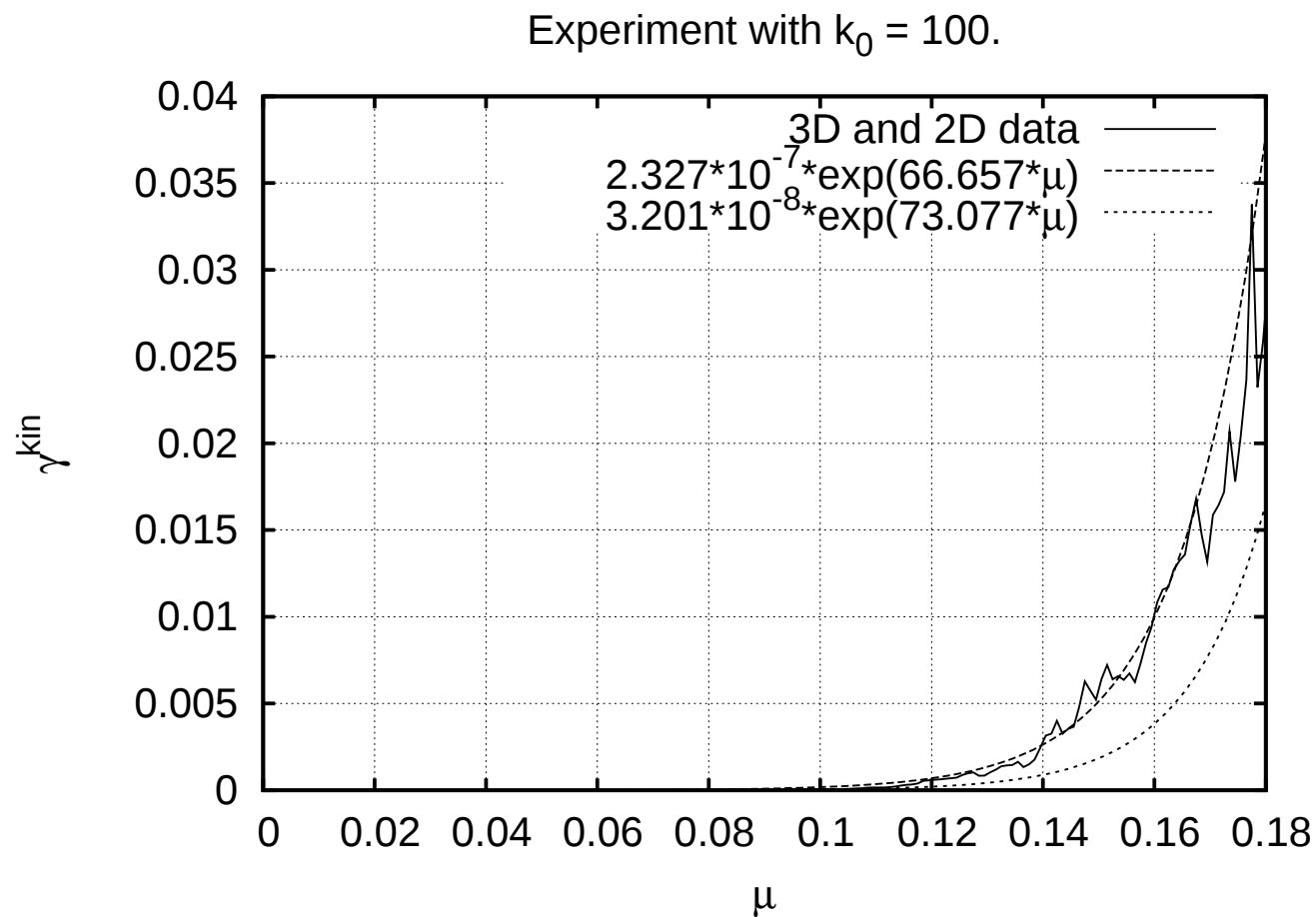
and for the *WAM4* case are:

$$C_{ds} = 4.09 \times 10^{-5}, \quad \delta = 0.5, \quad p = 4 \quad (18)$$

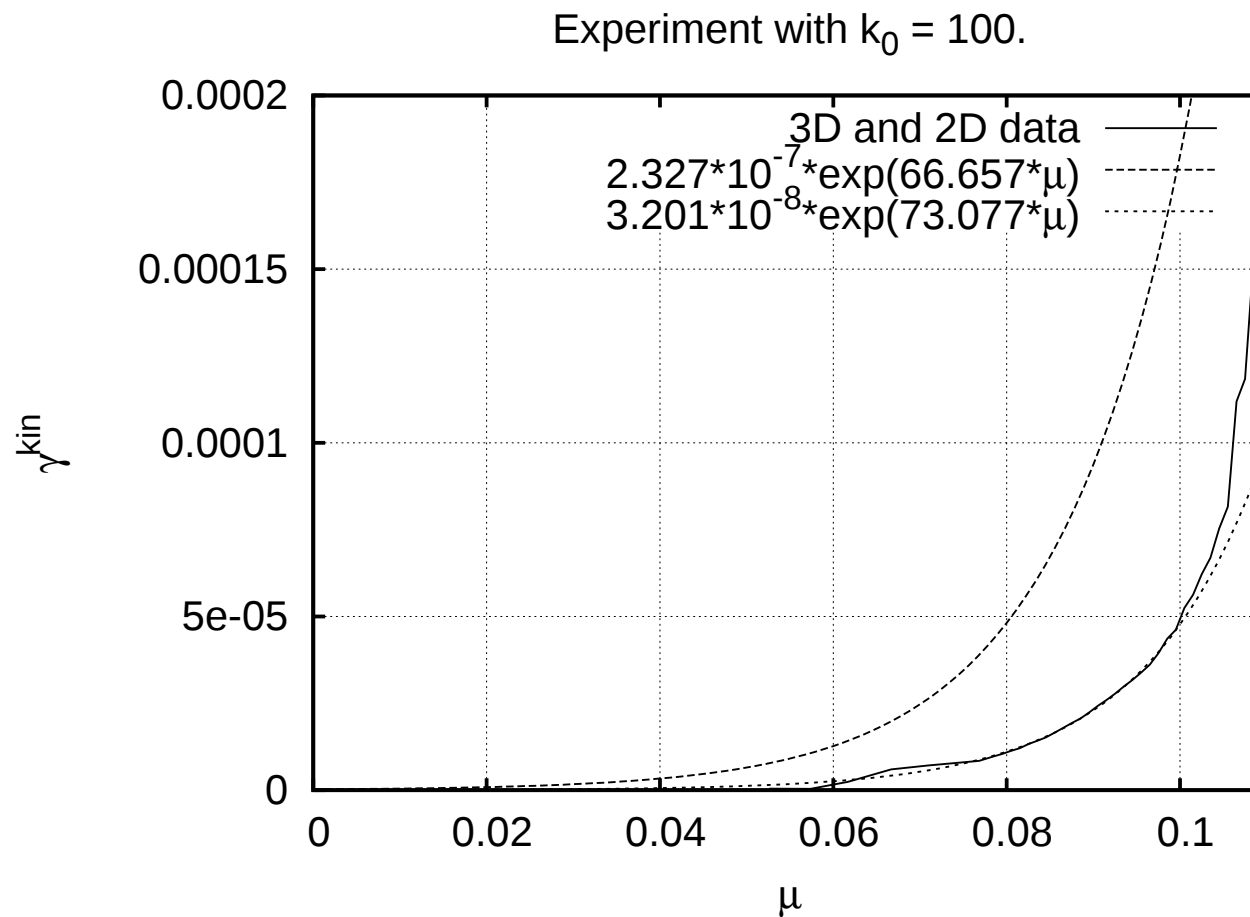
Dissipation function. Comparison with waves forecasting models.



Dissipation function. Exponential fit.



Dissipation function. Exponential fit. Low μ .



Results.

- Performed simulation of the gravity waves decaying turbulence in 2D fully-nonlinear and 3D weakly-nonlinear models.
- Obtained dependence of the dissipation function on average steepness.
- Demonstrated threshold-like character of the dissipation due to whitecapping.
- Results are significantly different with respect to wave-forecasting models terms.