Dark Solitons & Vortices in Bose-Einstein Condensates

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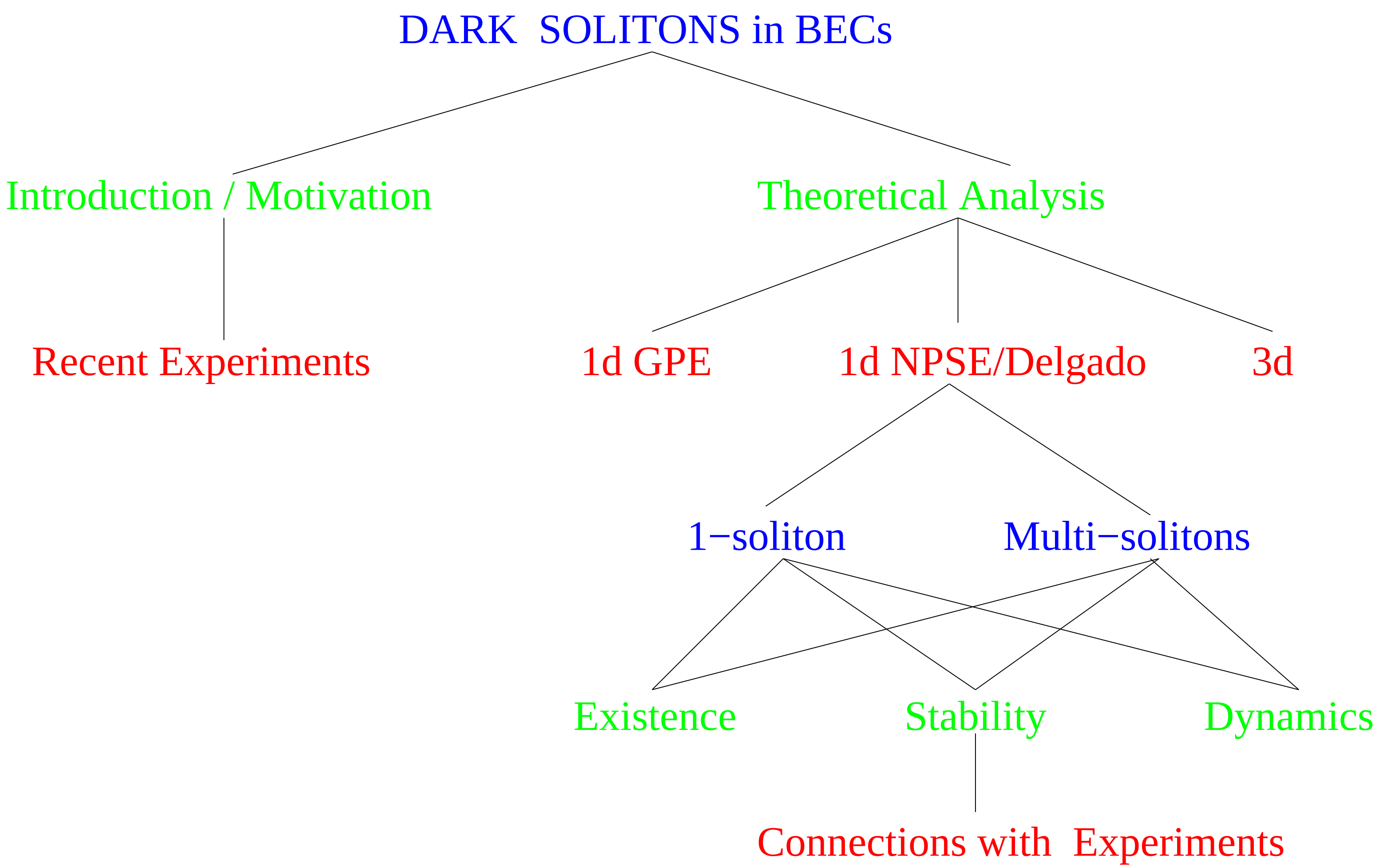
References

- 1d GPE:
  - Contemporary Mathematics **473**, 159 (2008)
  - Z. Agnew. Math. Phys. **59**, 559 (2008)
  - arXiv:0909.1660
  - arXiv:0910.5249
  - arXiv:1002.0552
- 1d NPSE/Delgado $\rightarrow$ 3d Dynamics
  - Phys. Rev. A **76**, 045601 (2007)
- Experimental Results:
  - Phys. Rev. Lett. **101**, 130401 (2008)
  - arXiv:0909.2122
- Generalizations to Vortices:
  - arXiv:0911.3308

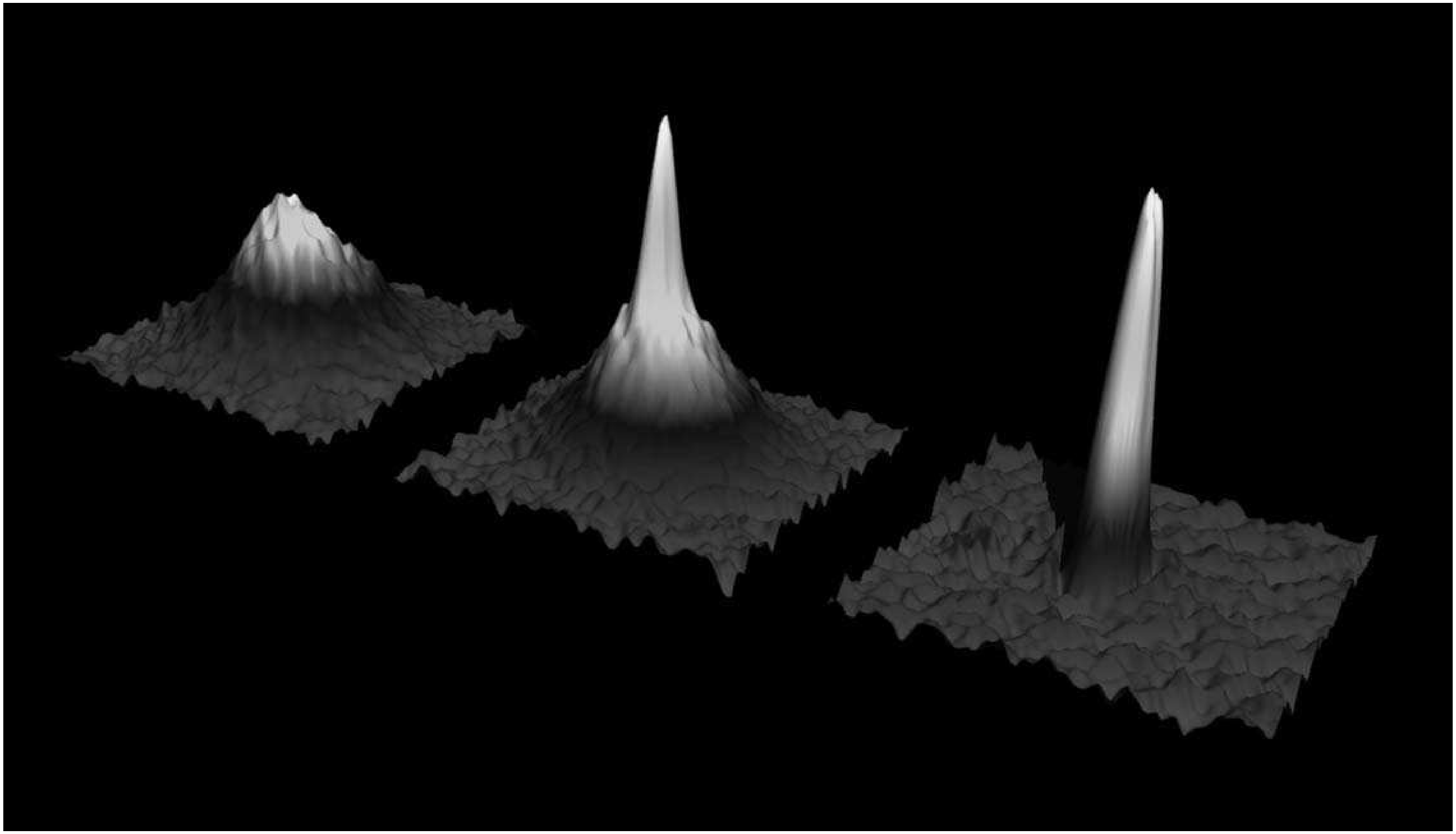

DARK SOLITONS in BECs
Introduction / Motivation
Theoretical Analysis
Recent Experiments
1d GPE
1d NPSE/Delgado
3d
1−soliton
Multi−solitons
Existence
Stability
Dynamics
Connections with Experiments

Brief Introduction to BECs

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States: a Quantum Phase Transition.

- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.

- Today:
  - ~ 35 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^3) Theoretical and O(10^2) Experimental papers ! Check out: http://amo.phy.gasou.edu/bec.html/bibliography.html



Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\widehat{H} = \int d\mathbf{r}\widehat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\mathbf{r})\right]\widehat{\Psi} + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\widehat{\Psi}^\dagger(\mathbf{r})\widehat{\Psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\widehat{\Psi}(\mathbf{r}')\widehat{\Psi}(\mathbf{r}) \tag{1}$$

- Bogoliubov Decomposition:

$$\widehat{\Psi} = \Phi(\mathbf{r}, t) + \widehat{\Psi}'(\mathbf{r}, t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + V_{\text{ext}}(\mathbf{r})\Phi + g|\Phi|^2\Phi \tag{3}$$

- for dilute, cold, binary collision gas.

- But: This is 3D NLS with a Potential: GP !

Low Dimensional Reductions

- 1d Magnetic Trap and/or Optical Lattice

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \sin^2(kx + \theta) \tag{4}$$

- 2d Magnetic Trap and/or Optical Lattice

$$V(x, y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + V_0\left(\sin^2(kx + \theta) + \sin^2(ky + \theta)\right) \tag{5}$$

- Discrete Models: Reduction through Wannier functions (WF)

$$\psi(x, t) = \sum_{n\alpha} c_{n,\alpha}(t) w_{n,\alpha}(x) \tag{6}$$

where the WF w_n of band α are expressed in terms of the Bloch functions $\phi_{k,\alpha}$ as:

$$w_\alpha(x - nL) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \varphi_{k,\alpha}(x) e^{-inkL} dk. \tag{7}$$

- Then the GP equation becomes into a Vector Lattice equation:

$$i\frac{dc_{n,\alpha}}{dt} = \widehat{\omega}_{0,\alpha} c_{n,\alpha} + \widehat{\omega}_{1,\alpha}\left(c_{n-1,\alpha} + c_{n+1,\alpha}\right) + s \sum_{\alpha_1,\alpha_2,\alpha_3} W^{nnnnn}_{\alpha\alpha_1\alpha_2\alpha_3} c^*_{n,\alpha_1} c_{n,\alpha_2} c_{n,\alpha_3} \qquad (8)$$

- which in the Tight Binding, Single Band Limit becomes the DNLS

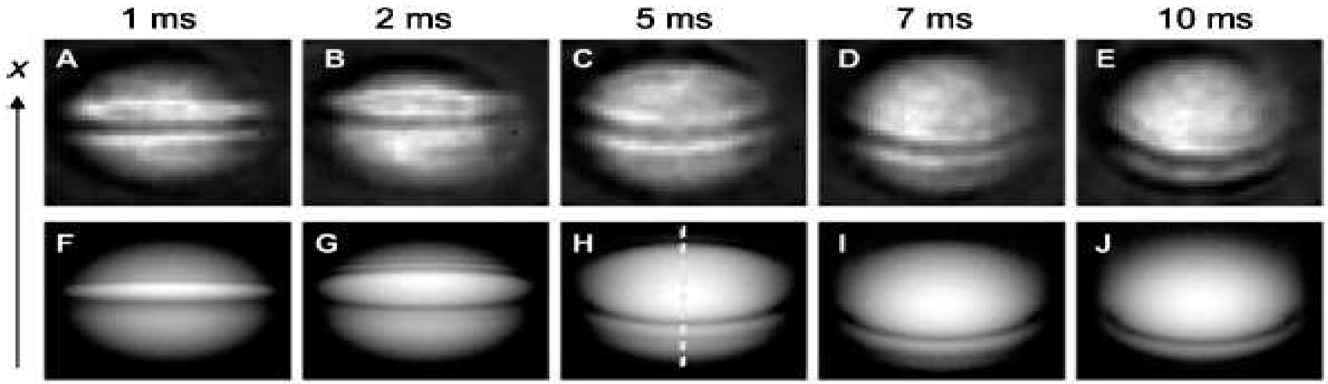
$$i\frac{dc_{n,\alpha}}{dt} = \widehat{\omega}_{0,\alpha} c_{n,\alpha} + \widehat{\omega}_{1,\alpha}\left(c_{n-1,\alpha} + c_{n+1,\alpha}\right) + sW^{nnnn}_{1111} |c_{n,\alpha}|^2 c_{n,\alpha}, \qquad (9)$$

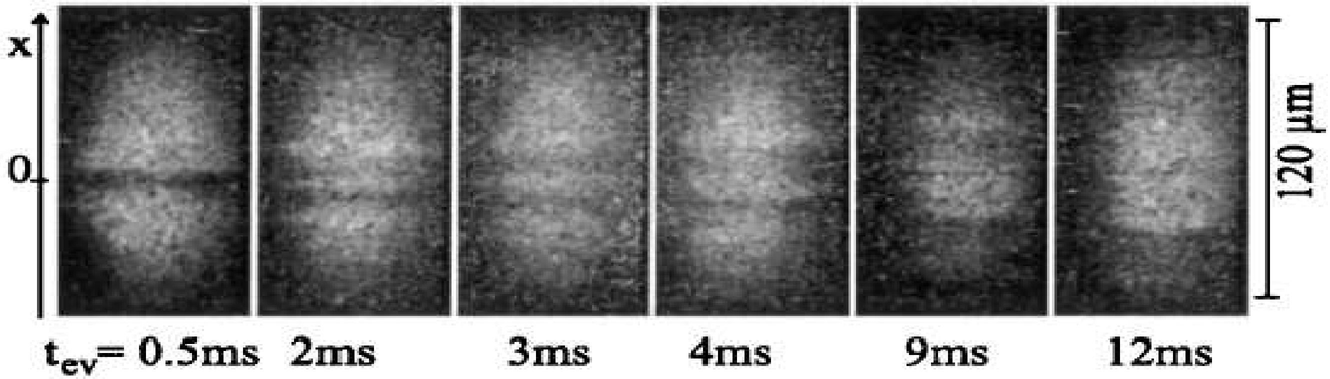
- and more generally the Vector Lattice Model with XPM

$$i\frac{d\tilde{c}_{n,\alpha}}{dt} = \widehat{\omega}_{1,\alpha}(\tilde{c}_{n-1,\alpha} + \tilde{c}_{n+1,\alpha}) + s\sum_{\alpha_1} W_{\alpha\alpha_1} |\tilde{c}_{n,\alpha_1}|^2 \tilde{c}_{n,\alpha}. \qquad (10)$$

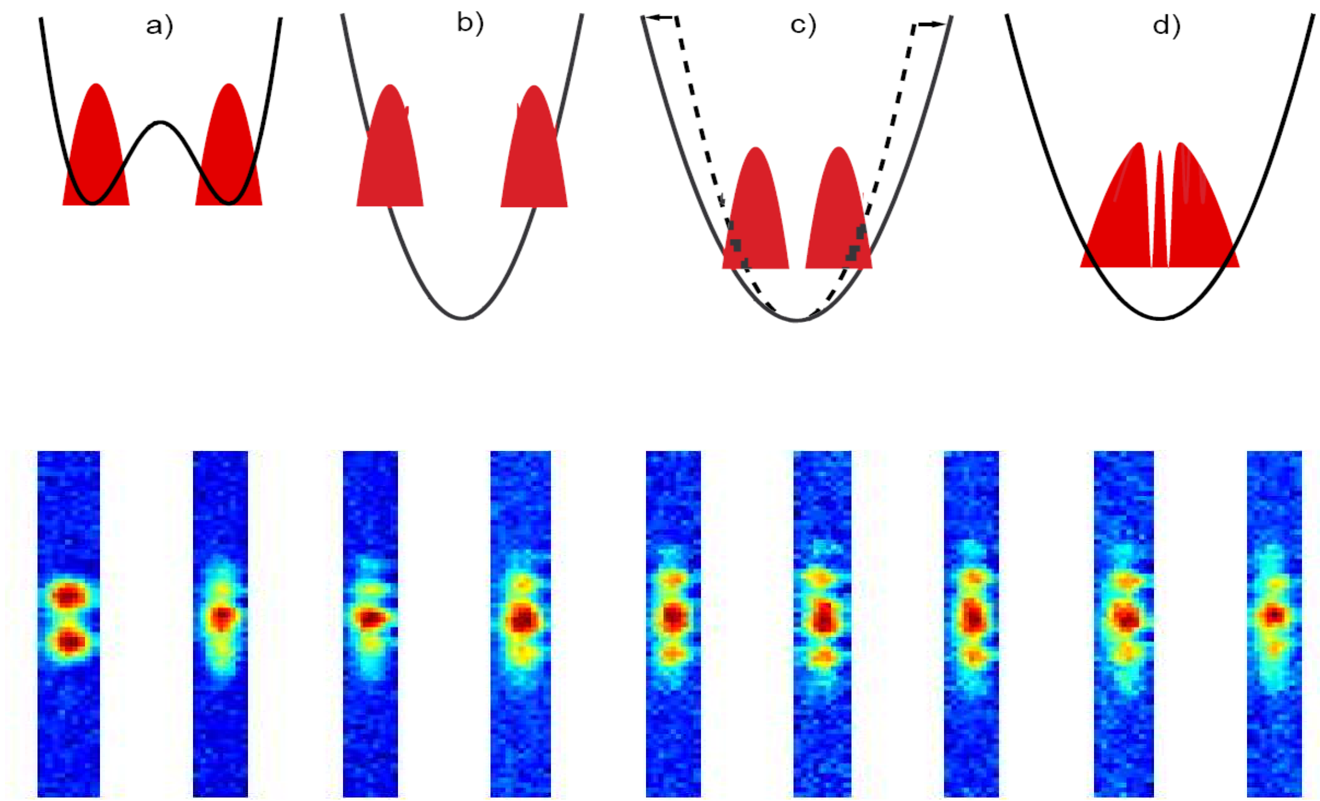
An Interesting Aspect: Dark Soliton Dynamics

Early Experiments

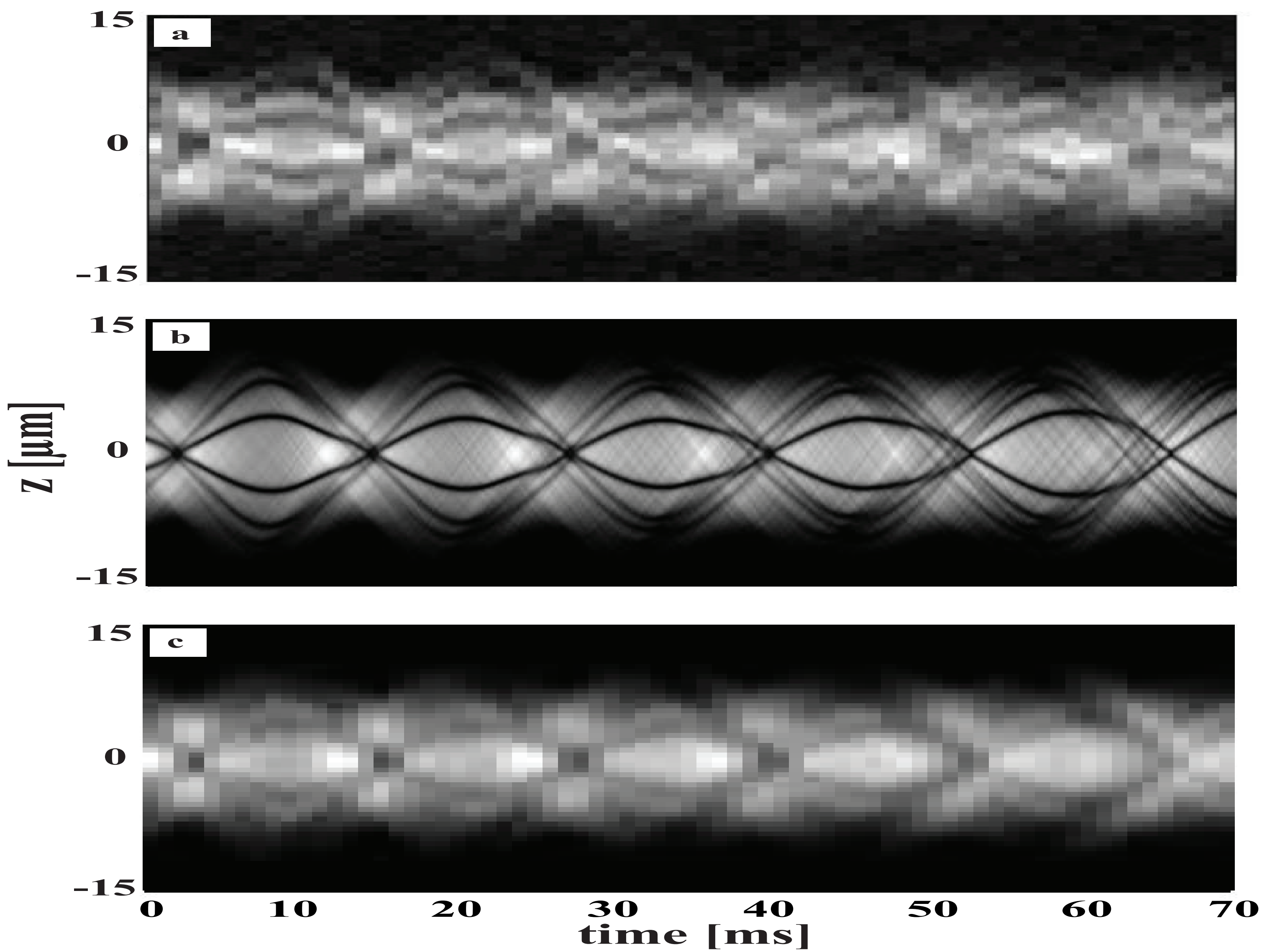




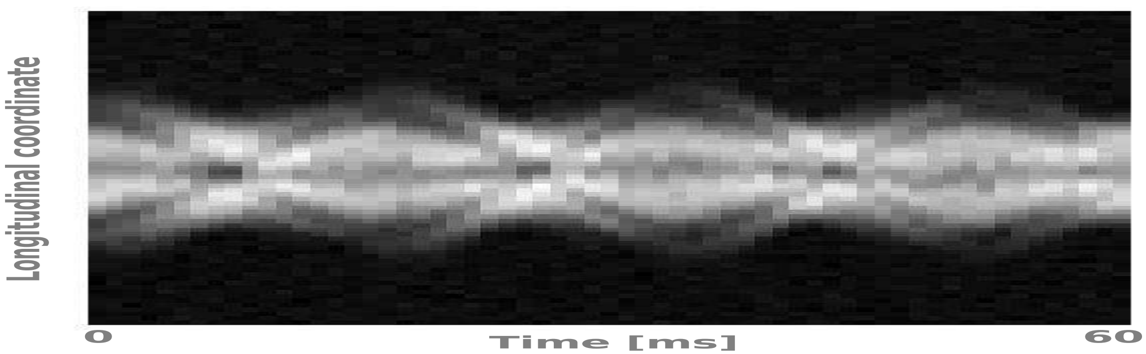
Our Specific Motivation: Dark Soliton Experiments in Heidelberg

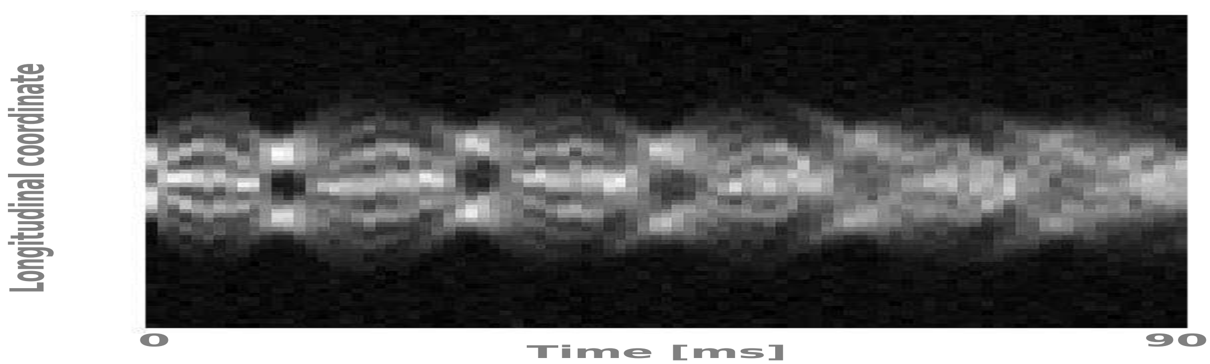


Detailed Dynamics and Computational Comparison



3-, 4-, N-soliton States





1st Line of Attack: 1d GPE

- Model reads

$$iu_t + \frac{1}{2}u_{xx} - |u|^2 u = \frac{1}{2}\omega^2 x^2 u \tag{11}$$

- Model assumes strong anisotropy ($\omega \ll 1$), and $\mu \ll \hbar\omega_\perp$, so that $\phi_0(r) \propto \exp(-r^2/2a_r^2)$.
- Consider Linear Limit $u(x,t) = \exp(-i\mu t)\sum_n A_n \phi_n(x)$ to obtain Bifurcation Function

$$F_n = (\mu - n)A_n - \sum_{n_1,n_2,n_3=0}^{\infty} K_{n,n_1,n_2,n_3} A_{n_1}\bar{A}_{n_2}A_{n_3}, \qquad \forall n = 0,1,2,... \tag{12}$$

- This leads to Near-Linear Solutions

$$\|\mathbf{A} - \varepsilon \mathbf{e}_1\|_{l^2_{1/2}} \le C_1 \varepsilon^3, \qquad \left|\mu - 1 - \sigma\varepsilon^2 K_{1,1,1,1}\right| \le C_2 \varepsilon^4, \tag{13}$$

where $K_{n,n_1,n_2,n_3} = (\phi_n, \phi_{n_1}\phi_{n_2}\phi_{n_3})$.

- Spectral Stability of These Solutions can be studied via:

$$\mathbf{a}(t) = e^{-i\mu t}\left[\mathbf{A} + (\mathbf{B} - \mathbf{C})\, e^{i\Omega t} + \left(\bar{\mathbf{B}} + \bar{\mathbf{C}}\right) e^{-i\bar{\Omega} t} + \mathrm{O}(\|\mathbf{B}\|^2 + \|\mathbf{C}\|^2)\right], \tag{14}$$

- This leads to Eigenvalue Problem:

$$L_+\mathbf{B} = \Omega \mathbf{C}, \qquad L_-\mathbf{C} = \Omega \mathbf{B}, \tag{15}$$

- In the above expression:

$$\left\{ \begin{array}{lcl} (L_+\mathbf{B})_n & = & (n-\mu)B_n + 3\sum_{m=0}^{\infty} V_{n,m}B_m, \\ (L_-\mathbf{C})_n & = & (n-\mu)C_n + \sum_{m=0}^{\infty} V_{n,m}C_m, \end{array} \right. \qquad \forall n = 0,1,2,3,..., \qquad (16)$$

where $V_{n,m} = \sum_{n_2,n_3=0}^{\infty} K_{n,m,n_2,n_3} A_{n_2} A_{n_3}$.

- Using Perturbation Theory, we obtain:

$$\left| \Omega_m - m + \varepsilon^2 \sigma \left(K_{1,1,1,1} - 2K_{m+1,1,1,m+1} \right) \right| \leq C_2 \varepsilon^4, \qquad (17)$$

$$\left| \Omega_1 - 1 + \frac{\varepsilon^2 \sigma}{8\sqrt{2\pi}} \right| \leq C_1 \varepsilon^4 \qquad (18)$$

- Another Limit known is the so-called Thomas-Fermi Limit

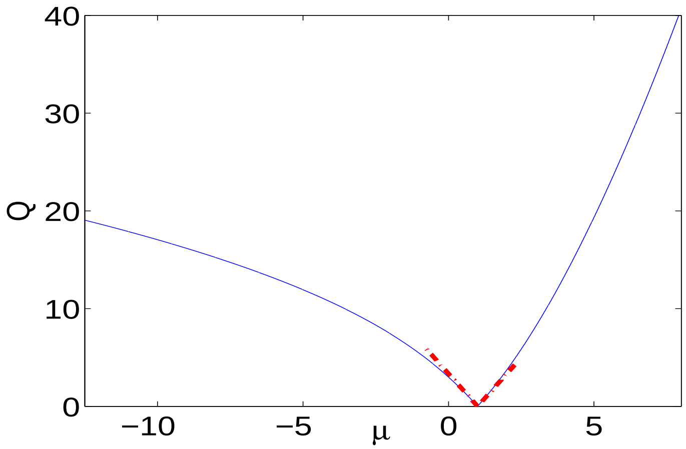
$$\sigma = 1: \quad \Omega_0 = 1, \ \lim_{\mu\to\infty} \Omega_1 = \frac{1}{\sqrt{2}}, \ \lim_{\mu\to\infty} \Omega_m = \frac{\sqrt{m(m+1)}}{\sqrt{2}}, \qquad m \geq 2 \qquad (19)$$

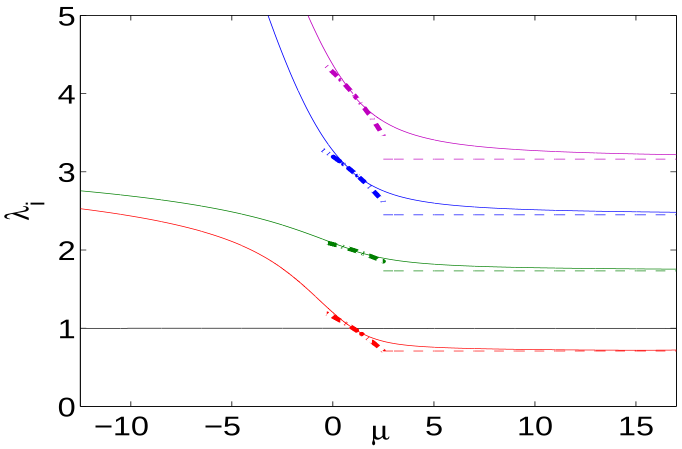
- Dipolar Oscillation Frequency $\Omega_0 = 1$ is fixed due to Transformation

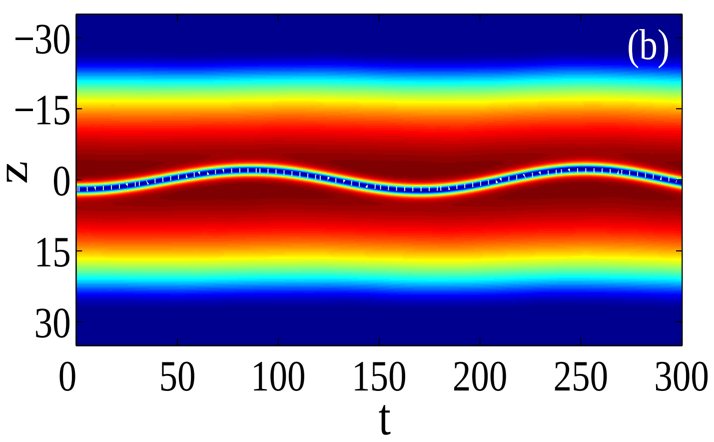
$$u(x,t) = e^{ip(t)x - \frac{i}{2}p(t)s(t) - \frac{i}{2}t - i\mu t - i\theta_0} \phi(x - s(t)), \qquad (20)$$

where $\dot{s} = p$, $\dot{p} = -s$

Numerical Findings in 1d Case







General Theory for Dark Solitons in 1d Case

- Consider the case of

$$iu_t = -\frac{1}{2}u_{xx} + |u|^2 u + \epsilon V(x) u \tag{21}$$

- Persistence Conditions for Dark Soliton $u = \phi_0(x-s)e^{-i\omega t + i\theta}$

$$M'(s_0) = \int_{\mathbb{R}} V'(x)\left[q_0 - \phi_0^2(x-s_0)\right] dx = 0 \tag{22}$$

$$M''(s_0) = \int_{\mathbb{R}} V''(x)\left[q_0 - \phi_0^2(x-s_0)\right] dx \neq 0. \tag{23}$$

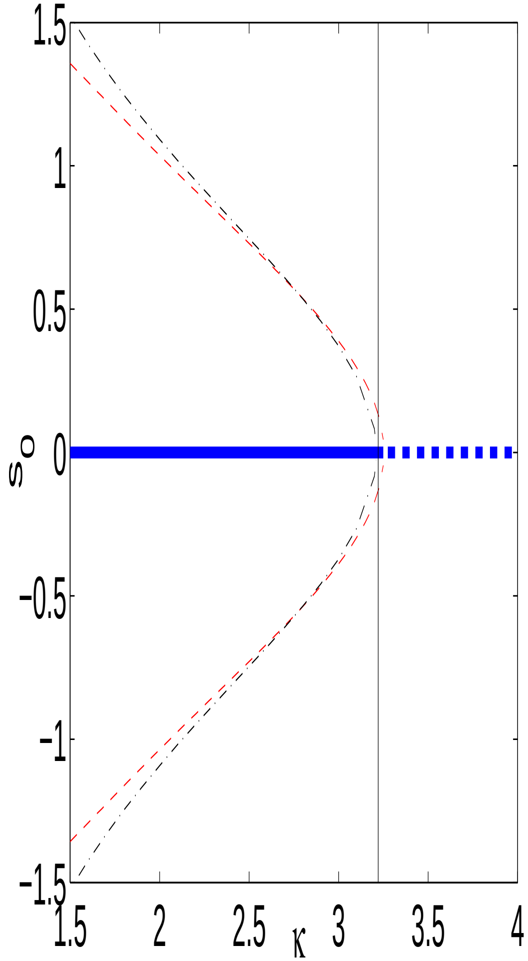
- Spectral Instability: real positive eigenvalue if $M''(s_0) < 0$ and two complex-conjugate eigenvalues with positive real part if $M''(s_0) > 0$.

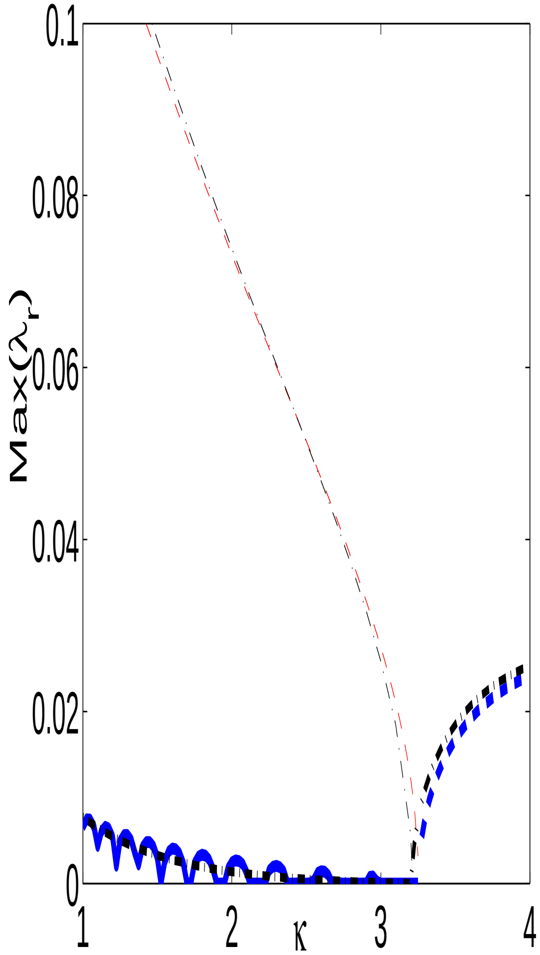
- The relevant Eigenvalues can be approximated as:

$$\mathrm{Re}\Lambda > 0 : \quad \Lambda^2 + \frac{\epsilon}{4}M''(s_0)\left(1 - \frac{\Lambda}{2}\right) = O(\epsilon^2). \tag{24}$$

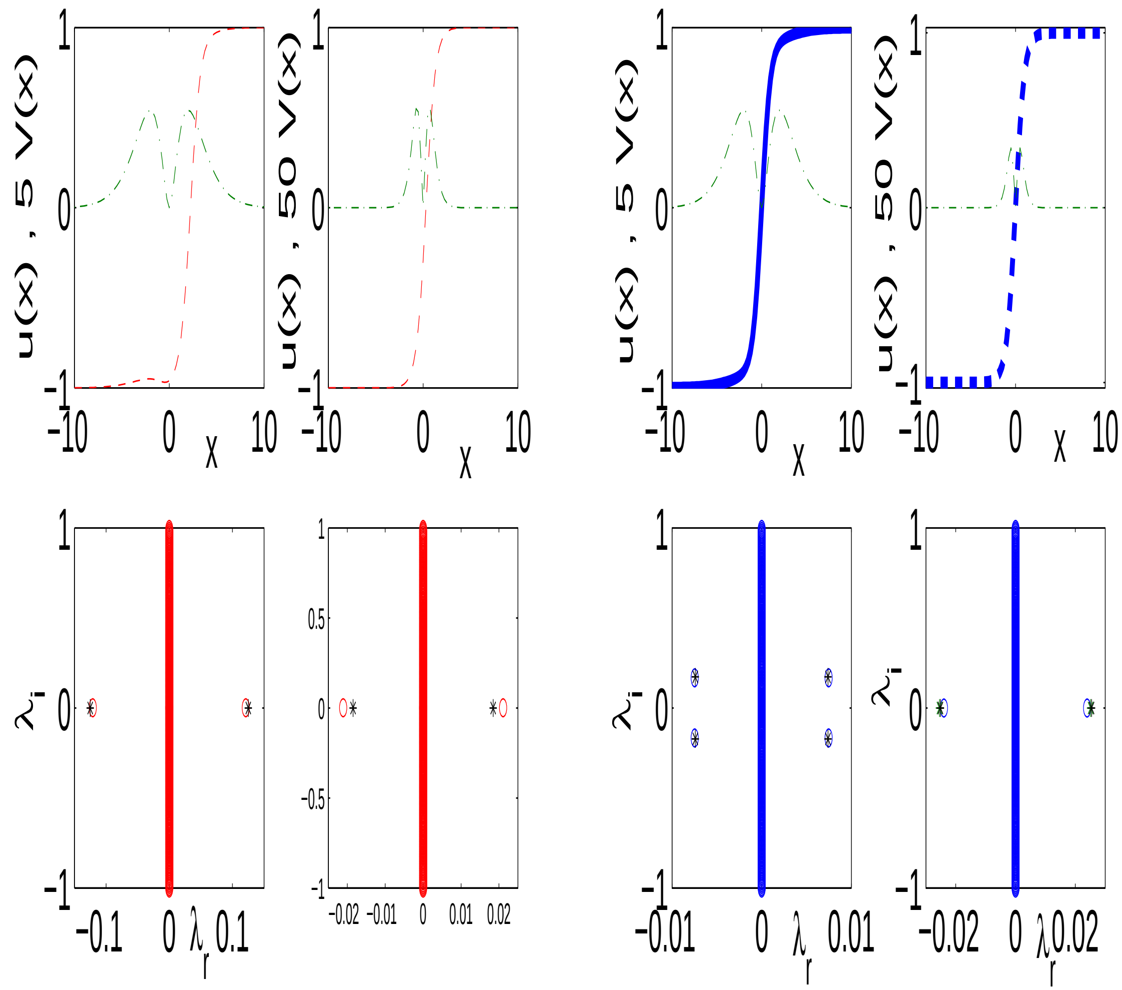
- This can be applied e.g. for $V(x) = x^2 e^{-\kappa|x|}$, but only formally for the case of $V(x) = \epsilon x^2$, where it gives $\Lambda^2 - \frac{\epsilon}{2}\Lambda + \epsilon = O(\epsilon^2)$.

Numerical Confirmation for Simpler Potentials





Numerical Confirmation for Simpler Potentials



2nd Line of Attack: Crossover Regime between 1d and 3d

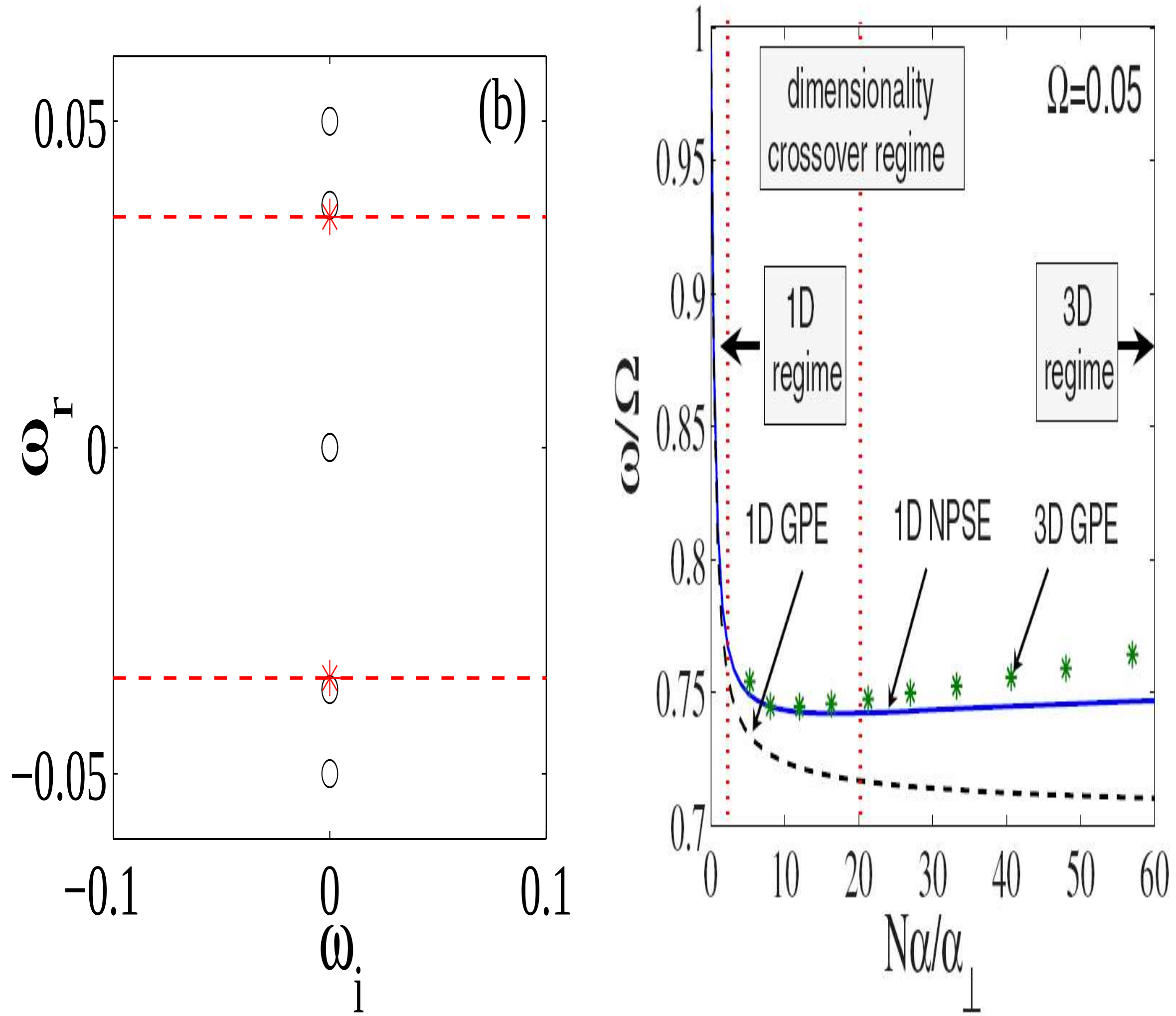
- Up to now $\omega \ll 1$, and $\mu \ll \hbar\omega_\perp$.
- However, realistic experiments are NOT in a quasi-1d regime of $N\omega a/a_\perp \ll 1$.
- They can be in so-called crossover regime of $\omega \ll 1$ and $\mu \approx \hbar\omega_\perp$ for which $N\omega a/a_\perp \simeq 1$ (Heidelberg) or in a truly 3d regime (Hamburg, Technion).
- For the crossover regime effective 1d equations have been obtained through variational principles extremizing the energy (Salasnich et al.) or the chemical potential (Delgado et al.) of the transverse degrees of freedom.
- For our purposes (time and space scales), these equations yield similar results, hence we will focus on the so-called NPSE (non-polynomial Schrödinger equation)

$$iu_t + \frac{1}{2}u_{xx} - \frac{3|u|^2+2}{2\sqrt{1+|u|^2}}u = \frac{1}{2}\omega^2 x^2 u \tag{25}$$

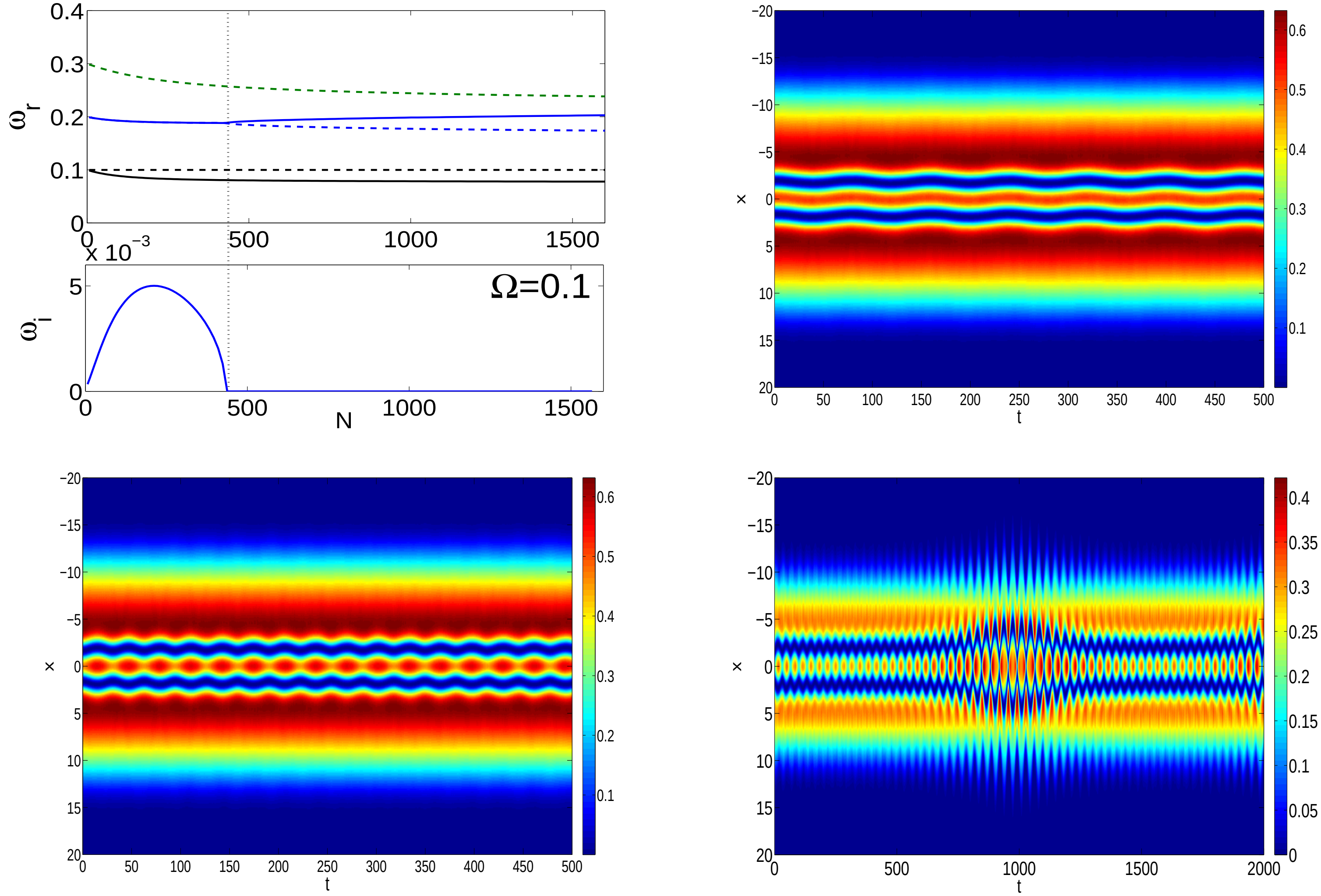
- Oscillation Frequency (anomalous mode) of Single Dark Soliton satisfies

$$\omega/\sqrt{2} < \omega_{1d}^{GPE} < \omega_{1d}^{NPSE} < \omega_{3d}^{GPE} \tag{26}$$

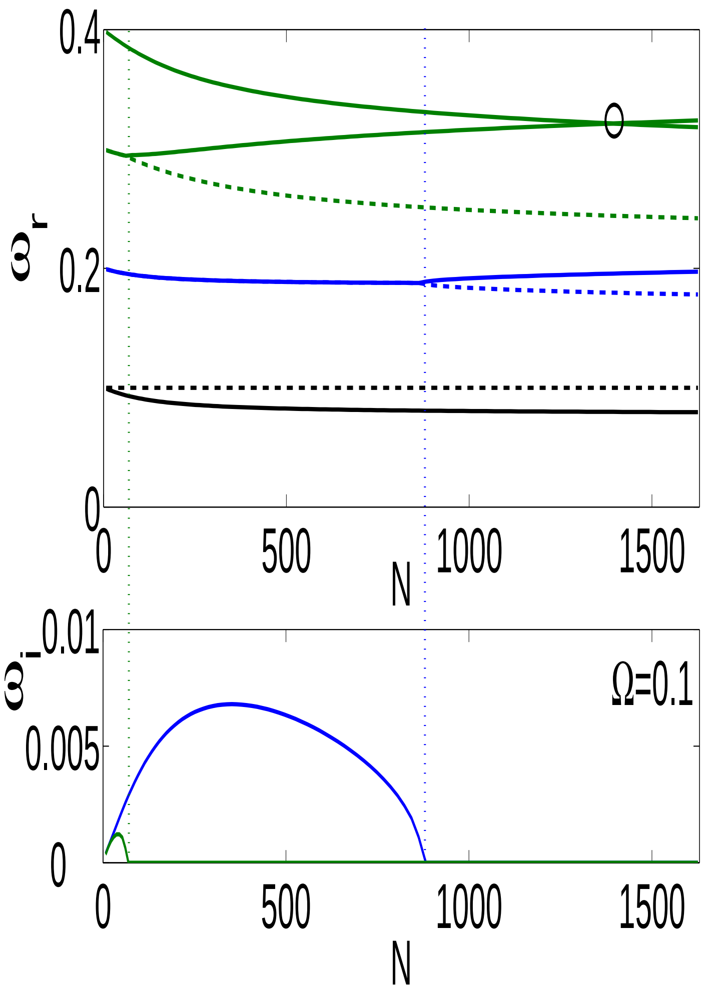
Single Dark Soliton Spectral Picture in NPSE

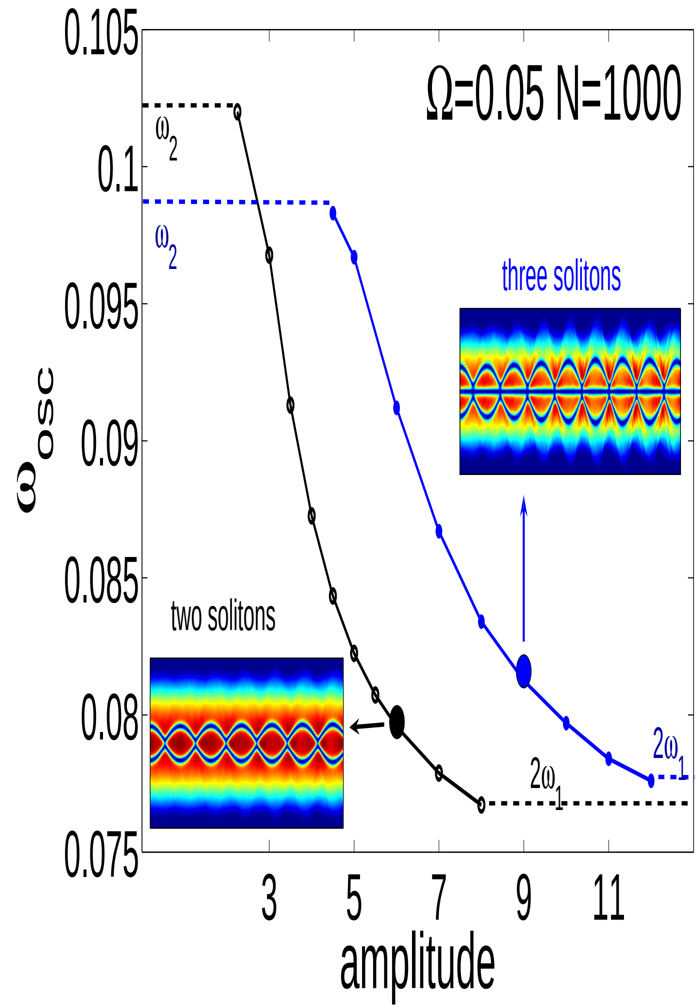


2-soliton Statics/Stability in the Crossover Regime

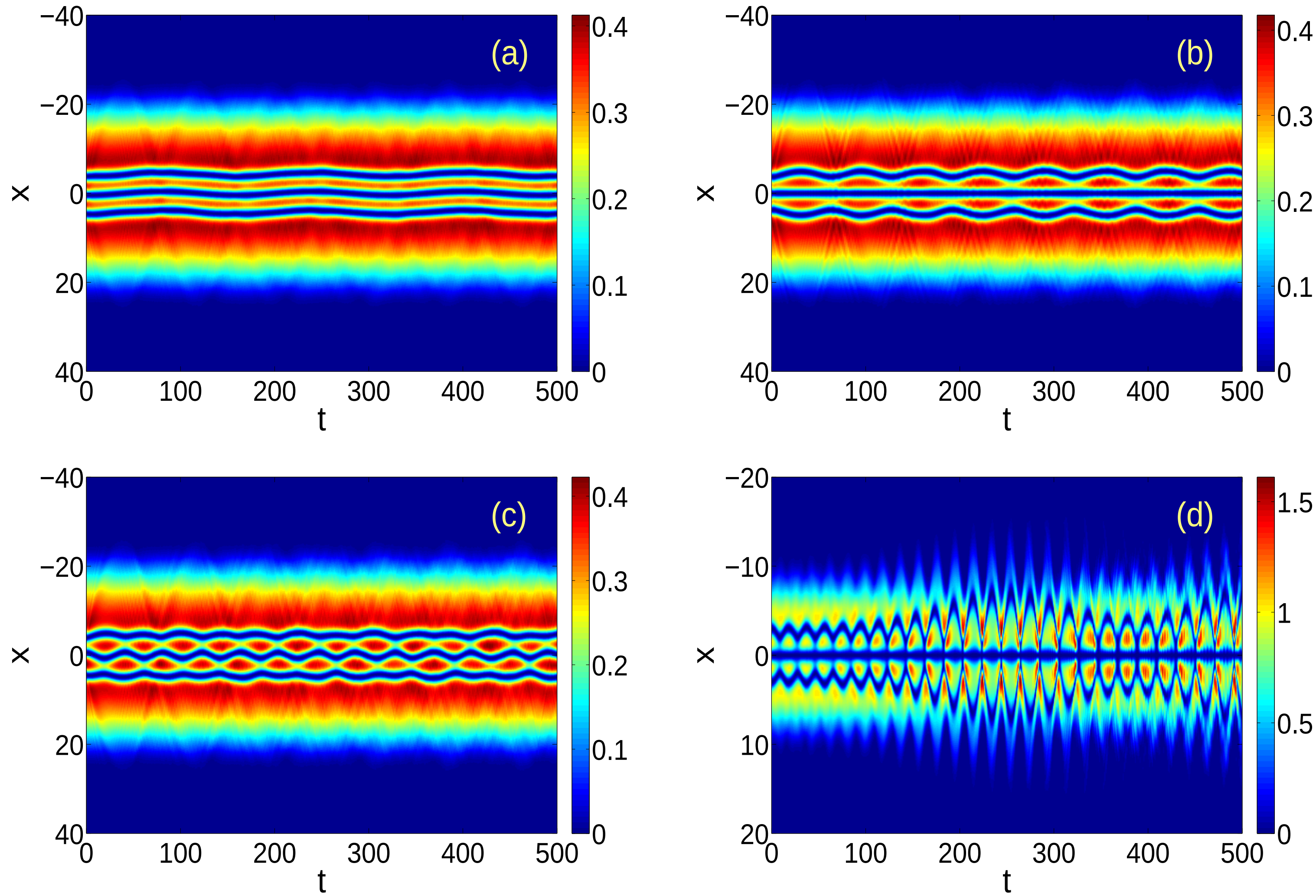


3-soliton Statics/Stability in the Crossover Regime





3-soliton Statics/Stability in the Crossover Regime (Continued)



3rd Line of Attack: Developing Soliton Dynamics

- Integrable 1d NLS has 2-soliton solution (Akhmediev et al.)

$$u(x,t) = \frac{(2a_3 - 4a_1)\cosh(\frac{\mu t}{2}) - 2\sqrt{a_1 a_3}\cosh(2px) + i\sinh(\frac{\mu t}{2})}{2\sqrt{a_3}\cosh(\frac{\mu t}{2}) + 2\sqrt{a_1}\cosh(2px)} e^{ia_3 t} \tag{27}$$

where $\mu = 4\sqrt{a_1(a_3 - a_1)}$ and $p = \sqrt{a_3 - a_1}$.

- By computing $\partial|u|^2/\partial x = 0$, we find soliton trajectories

$$\cosh(2px_0) = \sqrt{\frac{a_3}{a_1}}\cosh(\frac{\mu t}{2}) - 2\sqrt{\frac{a_1}{a_3}}\frac{1}{\cosh(\frac{\mu t}{2})} \tag{28}$$

- Interestingly, this yields insight on soliton interaction with point of closest approach

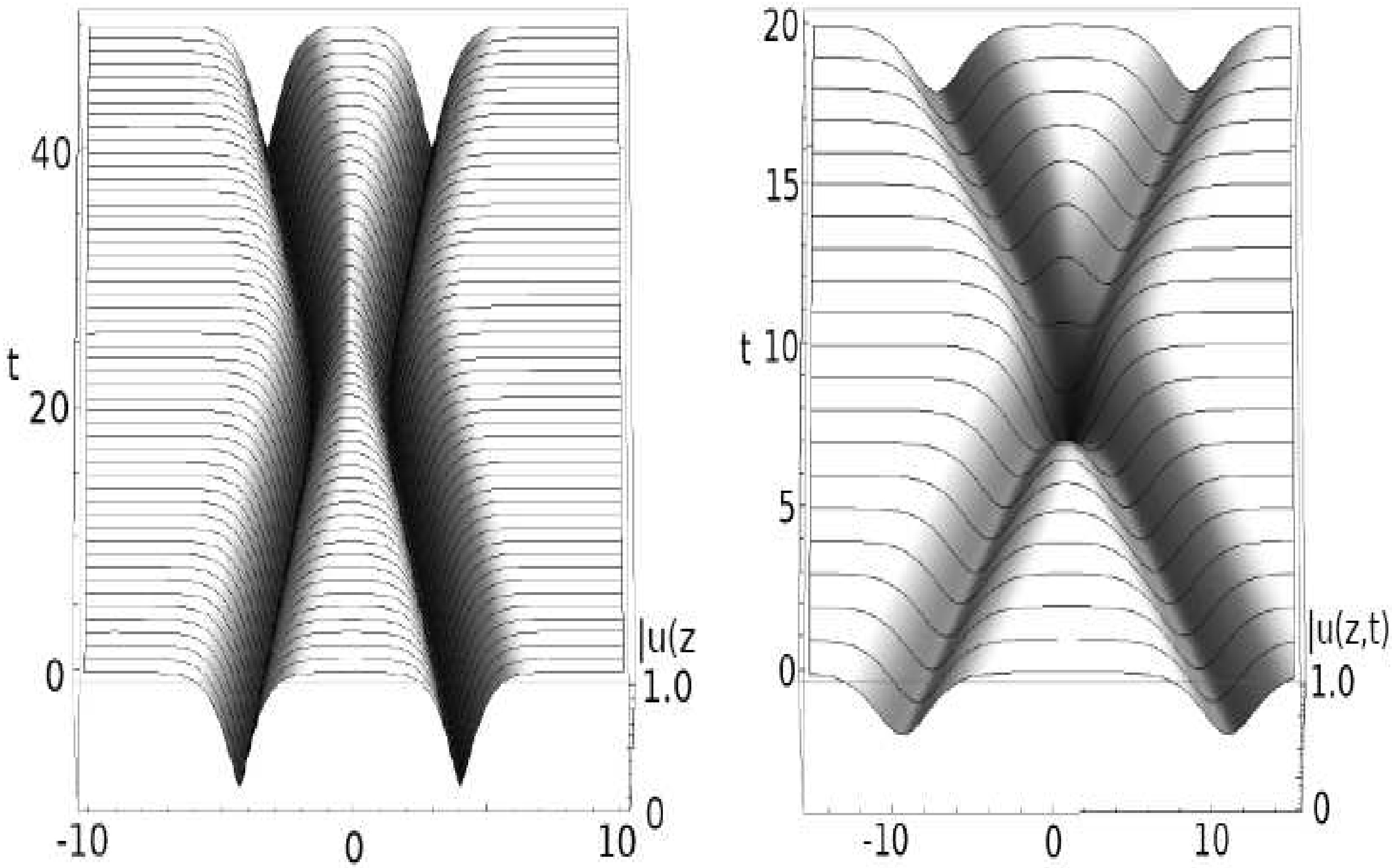
$$x_0 = \frac{1}{2\sqrt{a_3 - a_1}}\cosh^{-1}\left(\sqrt{\frac{a_3}{a_1}} - 2\sqrt{\frac{a_1}{a_3}}\right) \tag{29}$$

with $x_0 = 0$ for $a_1/a_3 = 1/4$ ($v = 0.5$)

- More importantly, for well-separated solitons (notice asymptotic limits)

$$\frac{dx_0}{dt} = \frac{\mu\sqrt{\frac{a_3}{a_1}}\sinh(\frac{\mu t}{2})}{2p\sqrt{\frac{a_3}{a_1}\cosh^2(\frac{\mu t}{2}) - 1}} \tag{30}$$

Integrable Dynamics



Developing Soliton Dynamics (With and Without Trap)

- Soliton Equation of Motion

$$\frac{d^2 x_0}{dt^2} = 8p^3 \frac{\cosh(2px_0)}{\sinh^3(2px_0)} \tag{31}$$

- Thus, obtain Inter-Soliton Interaction Potential (cf. Krolikowski-Kivshar)

$$V = \frac{1}{2} \frac{p^2}{\sinh^2(2px_0)} \tag{32}$$

- Importantly, notice that, in principle, V is dependent on $p = \sqrt{1 - \dot{x_0}^2}$.

- This expression can be generalized to Multi-Soliton Dynamics as:

$$V_{int} = \sum_{i \neq j}^{n} \frac{\mu p_{ij}^2}{2 \sinh^2[\sqrt{\mu} p_{ij}(x_i - x_j)]}. \tag{33}$$

- The Single Soliton Trapping Potential can be added:

$$V_{full} = V_{int} + \frac{1}{2}\omega_{eff}^2 x^2 \tag{34}$$

An Alternative View of this "Superposition"

4-th Line of Attack: Variational Approach for Dark Solitons

- For simplicity, consider the GPE model:

$$iv_\tau = -\frac{1}{2}v_{\xi\xi} + \frac{1}{2}\xi^2 v + |v|^2 v - \mu v, \tag{35}$$

- under the Change of Variables:

$$v(\xi, t) = \mu^{1/2} u(x,t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t, \tag{36}$$

leads to (with $\epsilon = (2\mu)^{-1}$ used as a Small Parameter)

$$i\epsilon u_t + \epsilon^2 u_{xx} + (1 - x^2 - |u|^2)u = 0, \tag{37}$$

- Then using information established about the ground state:

$$\eta_0(x) := \lim_{\epsilon \to 0} \eta_\epsilon(x) = \begin{cases} (1-x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1. \end{cases} \tag{38}$$

we can use the ansatz $u(x,t) = \eta_\epsilon(x) v(x,t)$ within the Lagrangian $L(v) = K(v) + \Lambda(v)$, where

$$\Lambda(v) = \epsilon^2 \int_{\mathbb{R}} \eta_\epsilon^2(x) |v_x|^2 dx + \frac{1}{2} \int_{\mathbb{R}} \eta_\epsilon^4(x)(1 - |v|^2)^2 dx. \tag{39}$$

$$K(v) = \frac{i}{2}\epsilon \int_{\mathbb{R}} \eta_\epsilon^2(x)(v\bar{v}_t - \bar{v}v_t) dx, \tag{40}$$

$1, 2, 3, \ldots \infty$: Towards a Lattice of Dark Solitons

Single Trapped Dark Soliton

- For a Single Dark Soliton (choosing $A = \sqrt{1 - b^2}$)

$$v_1(x,t) = A(t)\ \tanh\left(\epsilon^{-1} B(t)(x - a(t))\right) + ib(t), \tag{41}$$

the effective Lagrangian becomes

$$\begin{aligned} L_1 := \lim_{\epsilon \to 0} \frac{L(v_1)}{2\epsilon} = -\frac{\dot{b}}{\sqrt{1-b^2}}(a - \frac{1}{3}a^3) + b\sqrt{1-b^2}(1-a^2)\dot{a} \\ +\frac{2}{3}(1-a^2)(1-b^2)B + \frac{1}{3B}(1-a^2)^2(1-b^2)^2. \end{aligned}$$

This leads to:

$$B = \frac{\sqrt{1-a^2}\sqrt{1-b^2}}{\sqrt{2}}.$$

$$\dot{a} = \sqrt{2}\sqrt{1-a^2}b, \quad \dot{b} = -\frac{\sqrt{2}a(1-b^2)}{\sqrt{1-a^2}},$$

which, in turn, leads to:

$$\ddot{a} + 2a = 0.$$

- Now: 2-Soliton State (use $a_1 = -a_2 = -a$ and $b_1 = -b_2 = -b$):

$$v_2(x,t) = \left[A_1(t) \tanh\left(\epsilon^{-1}B_1(t)(x - a_1(t))\right) + ib_1(t)\right] \times \left[A_2(t) \tanh\left(\epsilon^{-1}B_2(t)(x - a_2(t))\right) + ib_2(t)\right], \quad (42)$$

yields the corresponding Lagrangian. After simplifications, one can write

$$\Lambda_2 := \frac{\Lambda(v_2)}{2\epsilon} = \Lambda_+ + \Lambda_- + \Lambda_{\text{overlap}}, \quad \Lambda_\pm = \frac{4(1-a^2)^{3/2}(1-b^2)^{3/2}}{3\sqrt{2}} + \mathcal{O}(\epsilon^{1/3}).$$

$$\Lambda_{\text{overlap}} = -8\sqrt{2}(1-a^2)^{3/2}(1-b^2)^{5/2}\, e^{-4Ba\epsilon^{-1}}\left(1 + \mathcal{O}(\epsilon^{1/3})\right).$$

- This, in turn, yields the nonlinear oscillator:

$$\ddot{a} + 2a = 8\sqrt{2}\epsilon^{-1}e^{-\frac{2\sqrt{2}a}{\epsilon}},$$

with equilibrium position:

$$a = \frac{\epsilon}{\sqrt{2}}\left(-\log(\epsilon) - \frac{1}{2}\log|\log(\epsilon)| + \frac{3}{2}\log(2) + o(1)\right) \quad \text{as} \quad \epsilon \to 0 \quad (43)$$

and oscillation frequency around it:

$$\omega_0^2(\epsilon) = 2 + \frac{32}{\epsilon^2}e^{-2\sqrt{2}a_0(\epsilon)\epsilon^{-1}} = 2 + \frac{4\sqrt{2}a_0(\epsilon)}{\epsilon} = -4\log(\epsilon) - 2\log|\log(\epsilon)| + 2 + 6\log(2) + o(1), \quad \text{as} \quad \epsilon \to 0. \quad (44)$$

m Trapped Dark Solitons

- Use the Ansatz:

$$v_m(x,t) = \prod_{j=1}^{m} \left(A_j(t)\ \tanh\left(\epsilon^{-1} B_j(t)(x - a_j(t))\right) + i b_j(t)\right). \tag{45}$$

- Obtain the Lagrangian

$$\text{Ł}_m \sim -\sqrt{2} \sum_{j=1}^{m} \left(a_j^2 + b_j^2\right) - 2 \sum_{j=1}^{m} a_j \dot{b}_j - 8\sqrt{2} \sum_{j=1}^{m-1} e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}}.$$

- Derive the Equations of Motion

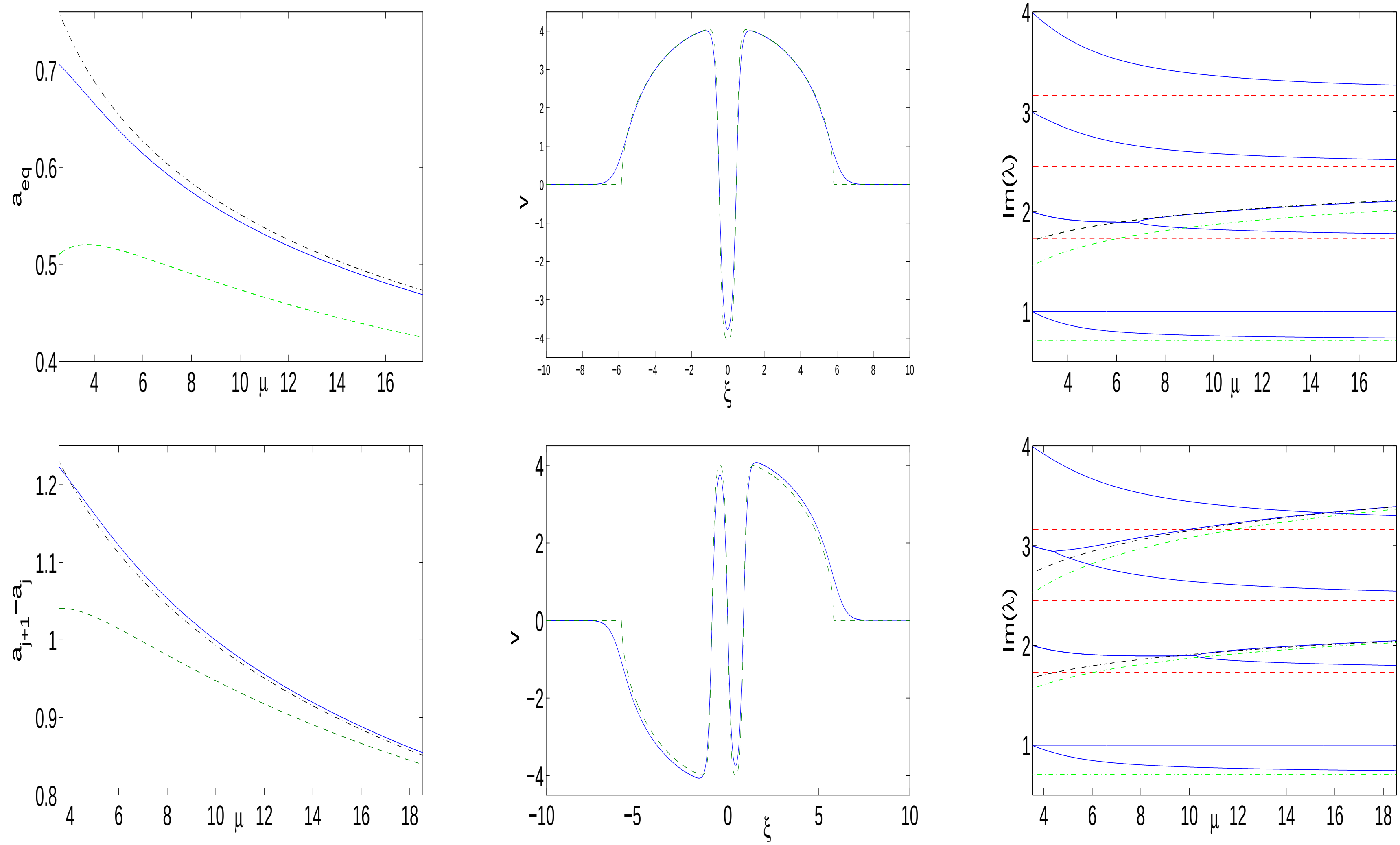
$$\dot{a}_j = \sqrt{2} b_j, \quad \dot{b}_j = -\sqrt{2} a_j - 8\epsilon^{-1} \left(e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}} - e^{-\sqrt{2}(a_j - a_{j-1})\epsilon^{-1}}\right). \tag{46}$$

- Define $x_j = \sqrt{2}(a_{j+1} - a_j)\epsilon^{-1}$, to find Equilibrium Distances and Oscillation Frequencies [With $\Omega^2 \in \left\{1, 3, 6, ..., \frac{m(m-1)}{2}\right\}$].

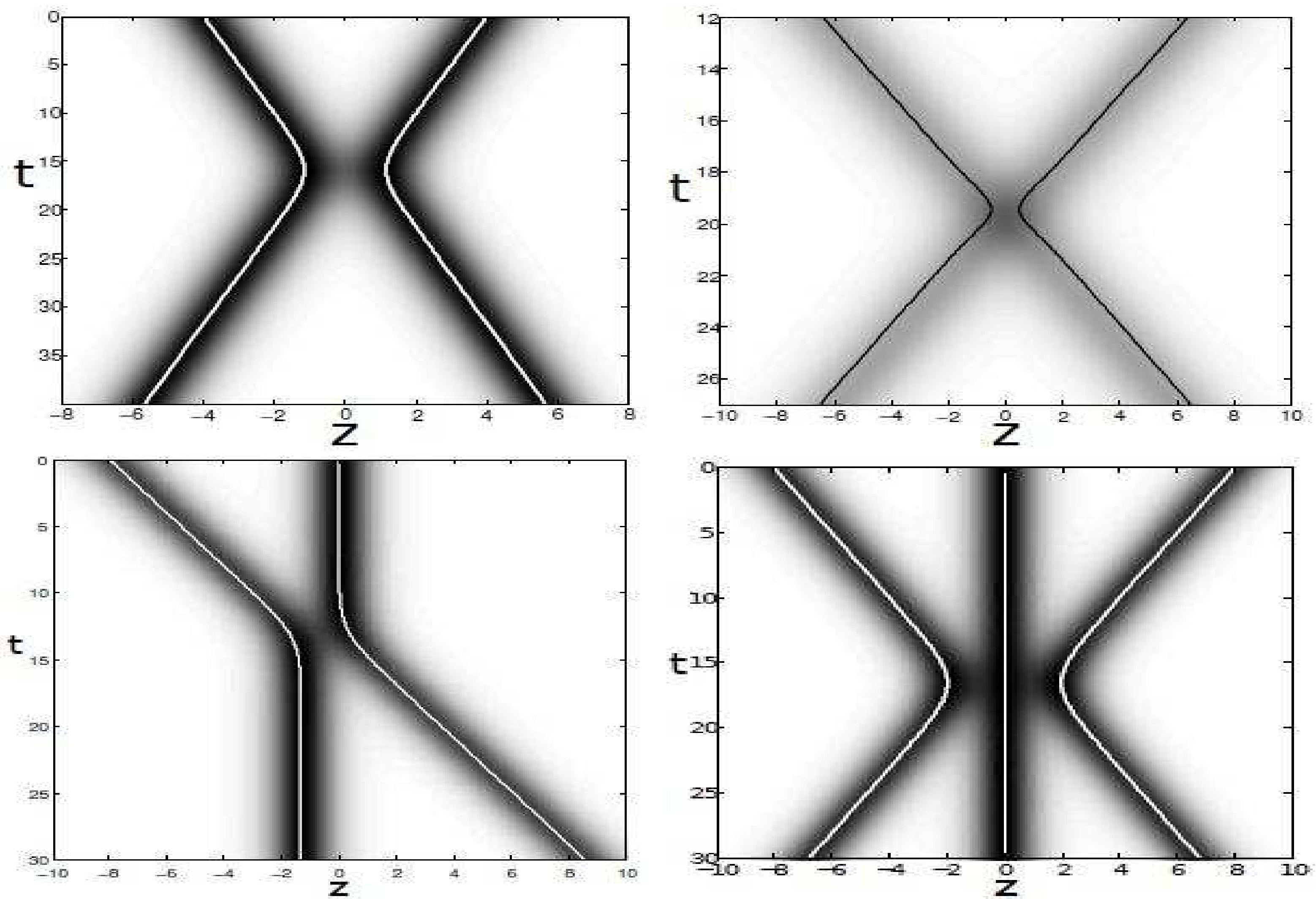
$$\mathbf{x} = -2\log(\epsilon)\mathbf{1} - \log|\log(\epsilon)|\mathbf{1} + 2\log(2)\mathbf{1} - \log(\mathbf{A}^{-1}\mathbf{1}) + o(1), \quad \text{as} \quad \epsilon \to 0, \tag{47}$$

$$\omega^2 = 2 + \left(-4\log(\epsilon) - 2\log|\log(\epsilon)| + 4\log(2)\right)\Omega^2 + \mathcal{O}(1). \tag{48}$$

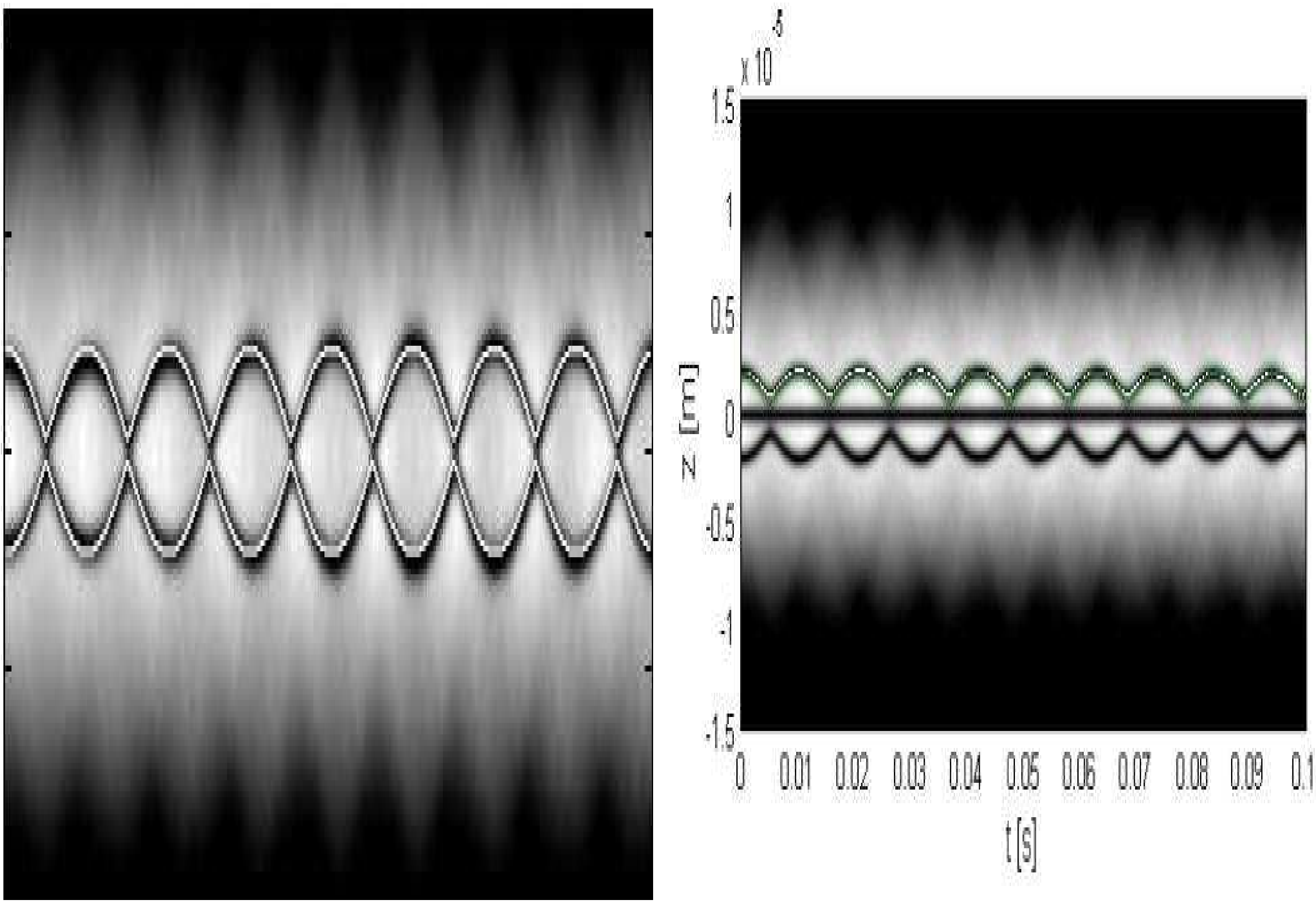
Testing the Prediction: Statics & Oscillation Frequencies



Testing the Prediction: Dynamics (Without Trapping)

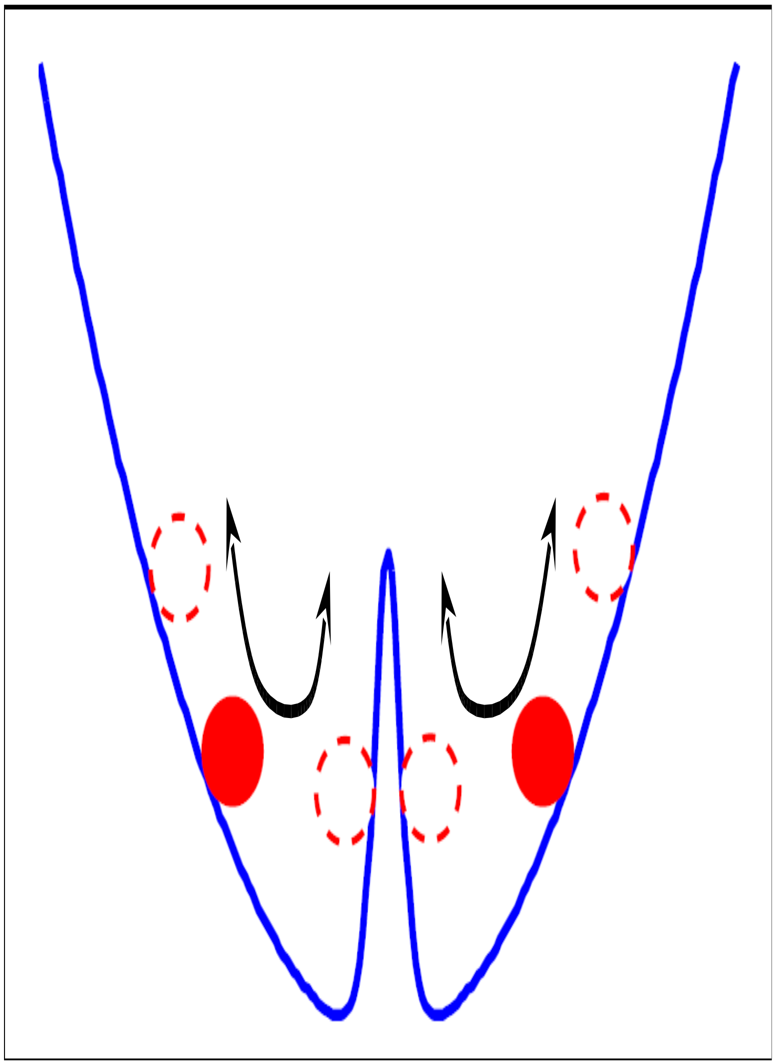


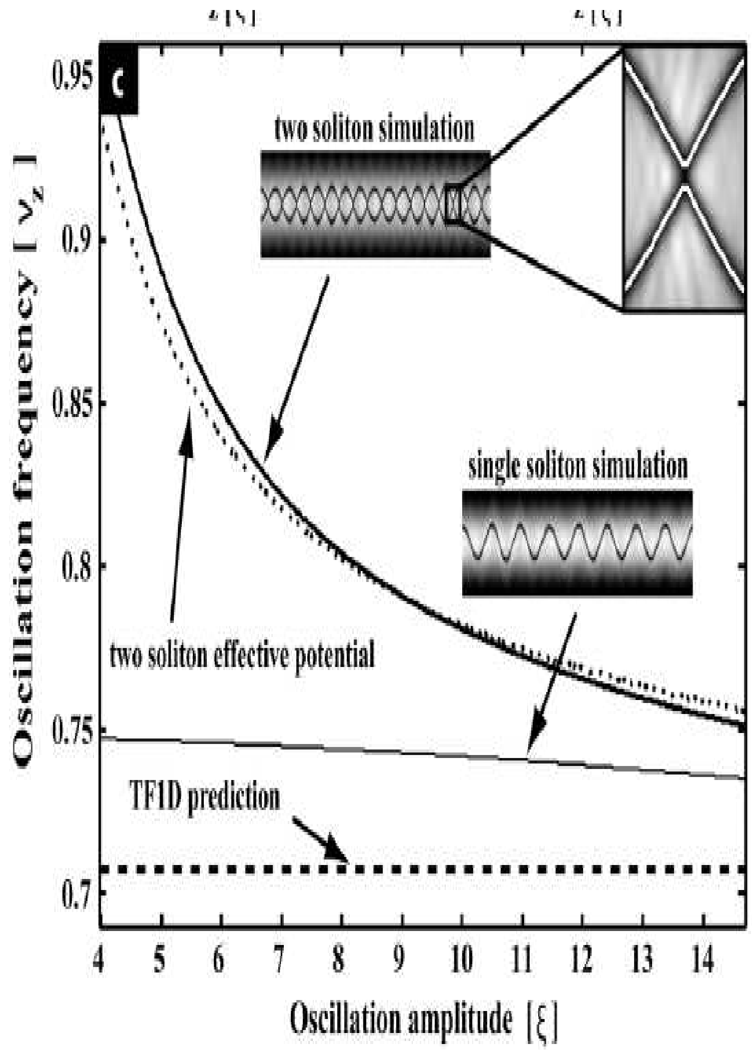
Testing the Prediction: Dynamics (With Trapping)



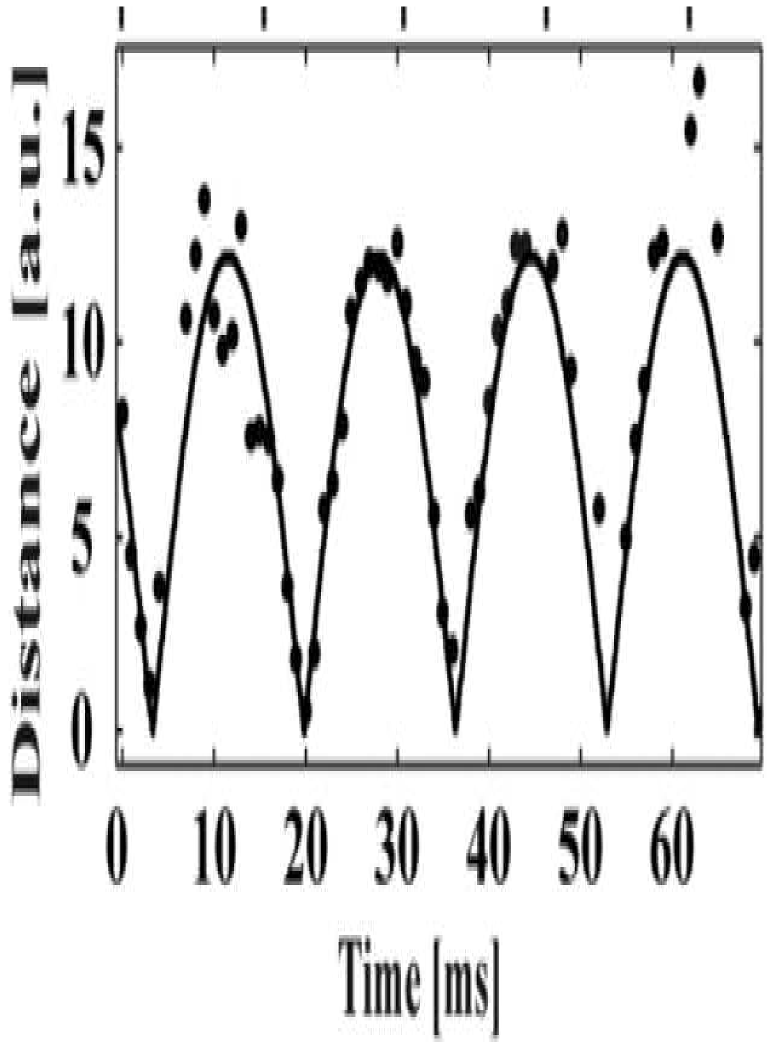
Now Collecting All the Chips !

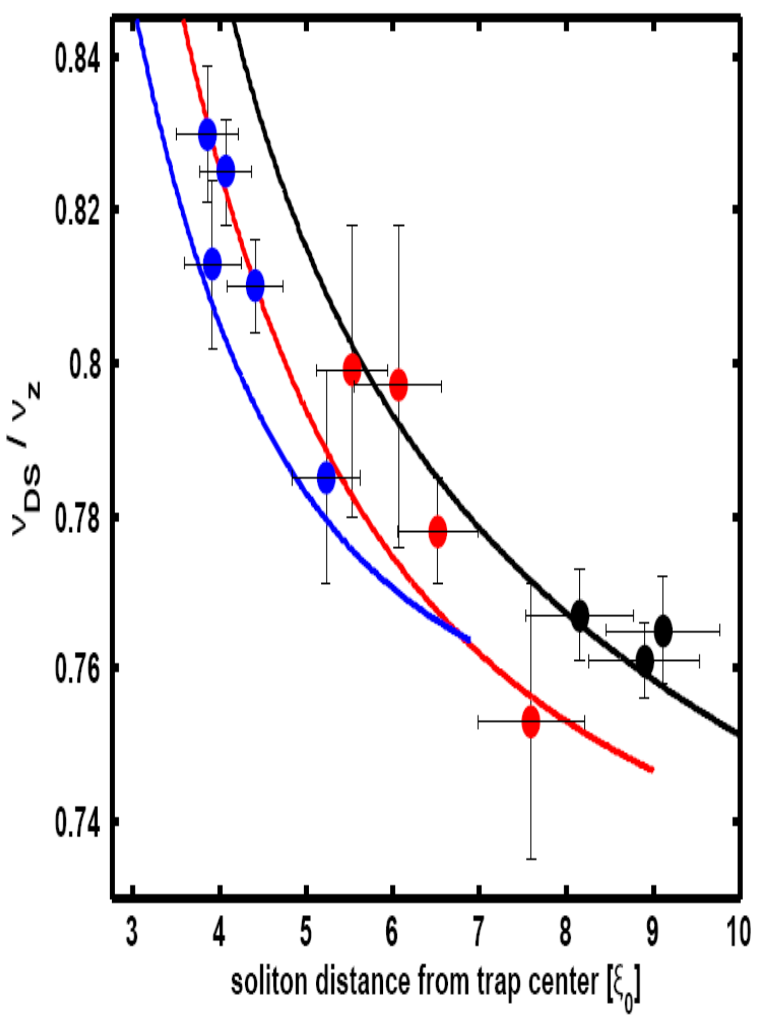
Effective Potential and its Oscillation Frequency Prediction





Comparison with Experiments





Summarizing the Picture

- First, characterize a Single Soliton and its Excitation Spectrum in the presence of External Potentials.
- Then, form a BEC in a Double Well Potential

$$V(x) = \frac{1}{2}\omega^2 x^2 + V_0 \cos^2(kx) \tag{49}$$

- Eliminate the Optical Lattice and Ramp Down the Magnetic Trap
- Observe the Formation and Characterize the Interaction Between the Dark Solitons.
- We obtain very good agreement between Experimental Results, Numerical (3d and 1d) Results and Analytical Results from a Simple Particle Picture

$$\frac{d^2x}{dt^2} = -\omega_{eff}^2\, x + \frac{B^3 \cosh(2xB)}{\sinh^3(2xB)}. \tag{50}$$

Present/Future Challenges

- Still, on Dark Solitons
  - Effects of Temperature $\longrightarrow$ Dissipative NLS Models
  - Generalization to N-Soliton States $\longrightarrow$ Dark Soliton Crystal vs. Dark Soliton Gas ?
- Dark Solitons, Take 2
  - Effects of Optical Lattice
  - Single & Multi-Soliton States
- Generalization to Vortices
  - Painting a Similar Picture $\longrightarrow$ Anomalous Modes, Precession, Characterization of Interactions
  - Generalization of the Picture $\longrightarrow$ Including Effects of Temperature, N-Vortex States
- Generalization to Multi-Components
  - Characterize Dark-Bright Soliton States $\longrightarrow$ Monitor their Oscillations and Interactions
  - Generalize these States $\longrightarrow$ Crystals, Thermal Effects ...

One Example From Each (Thermal Effects & Dark Solitons)

- Consider the Dissipative GPE as a way of modeling Finite Temperature Effects

$$(i-\gamma)\partial_t\psi = \left[\frac{1}{2}\partial_z^2 + V(z) + |\psi|^2 - \mu\right]\psi, \tag{51}$$

- Then, the motion of a Single Dark Soliton Characterized by:

$$\frac{d^2 z_0}{dt^2} = \left[\frac{2}{3}\gamma\mu\frac{dz_0}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 z_0\right] \cdot \left[1 - \left(\frac{dz_0}{dt}\right)^2\right]. \tag{52}$$

- At the Linearization level this yields:

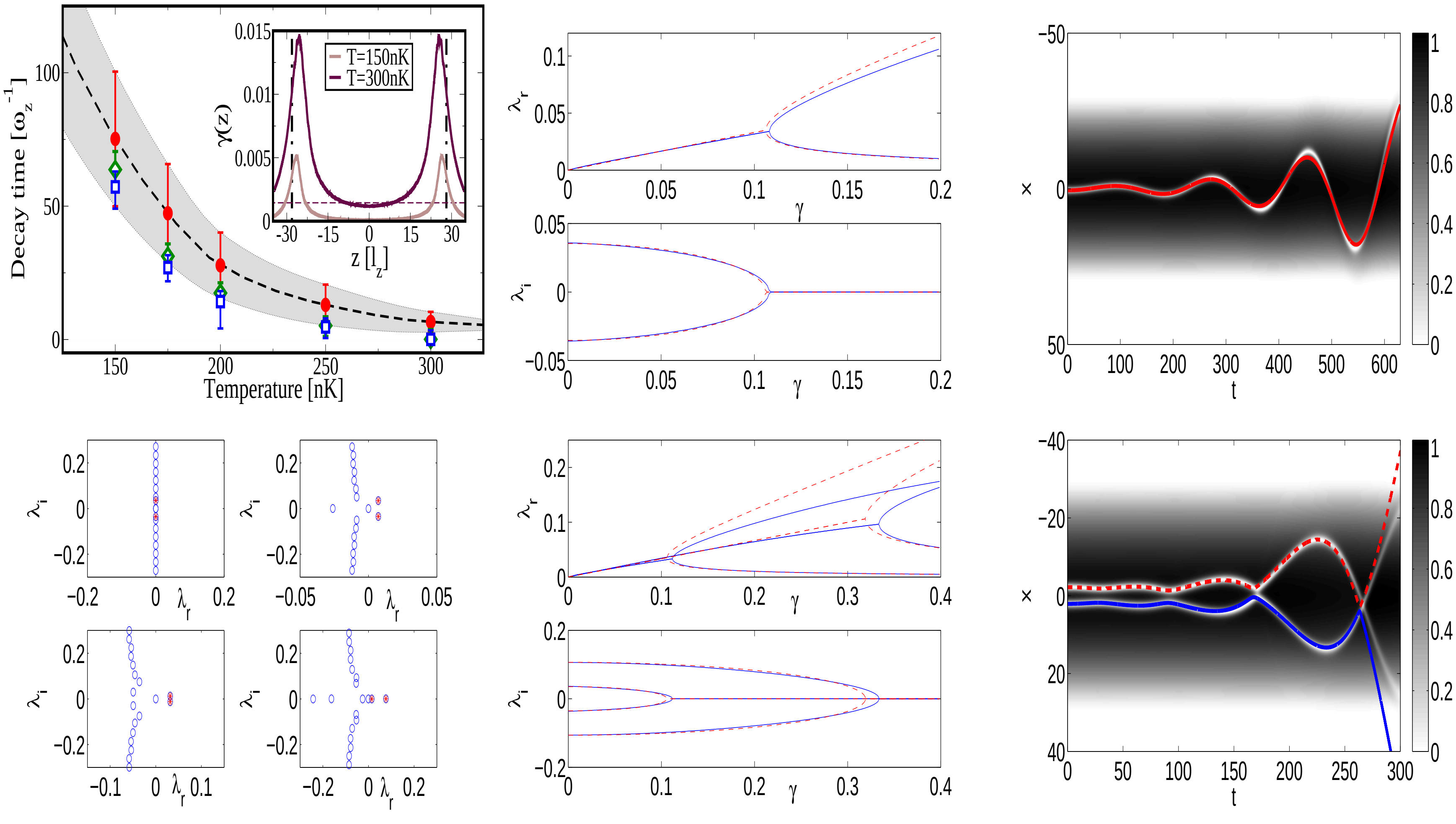
$$\omega_{1,2} = \frac{1}{3}\gamma\mu \pm \left(\frac{\Omega}{\sqrt{2}}\right)\sqrt{\Delta},$$

with $\Delta = \left(\frac{\gamma}{\gamma_{cr}}\right)^2 - 1$, and $\gamma_{cr} = (3/\mu)(\Omega/\sqrt{2})$.

- Also, this can be generalized to Multi-Soliton Dynamics e.g. for 2-solitons:

$$\frac{d^2 x_1}{dt^2} = \frac{2}{3}\gamma\frac{dx_1}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 x_1 - 8\exp\left(-(x_2 - x_1)\right), \tag{53}$$

$$\frac{d^2 x_2}{dt^2} = \frac{2}{3}\gamma\frac{dx_2}{dt} - \left(\frac{\Omega}{\sqrt{2}}\right)^2 x_2 + 8\exp\left(-(x_2 - x_1)\right). \tag{54}$$


Decay time [ω_z^{-1}]
Temperature [nK]
T=150nK
T=300nK
γ(z)
z [l_z]
λ_r
λ_i
γ
x
t

One Example From Each (Vortex Motion & Anomalous Mode)

- Vortices have Dynamics reminiscent of those of the Dark Solitons.
- They are dominated by an Oscillation Mode associated with their Precession Frequency. One can used Matched Asymptotics to obtain:

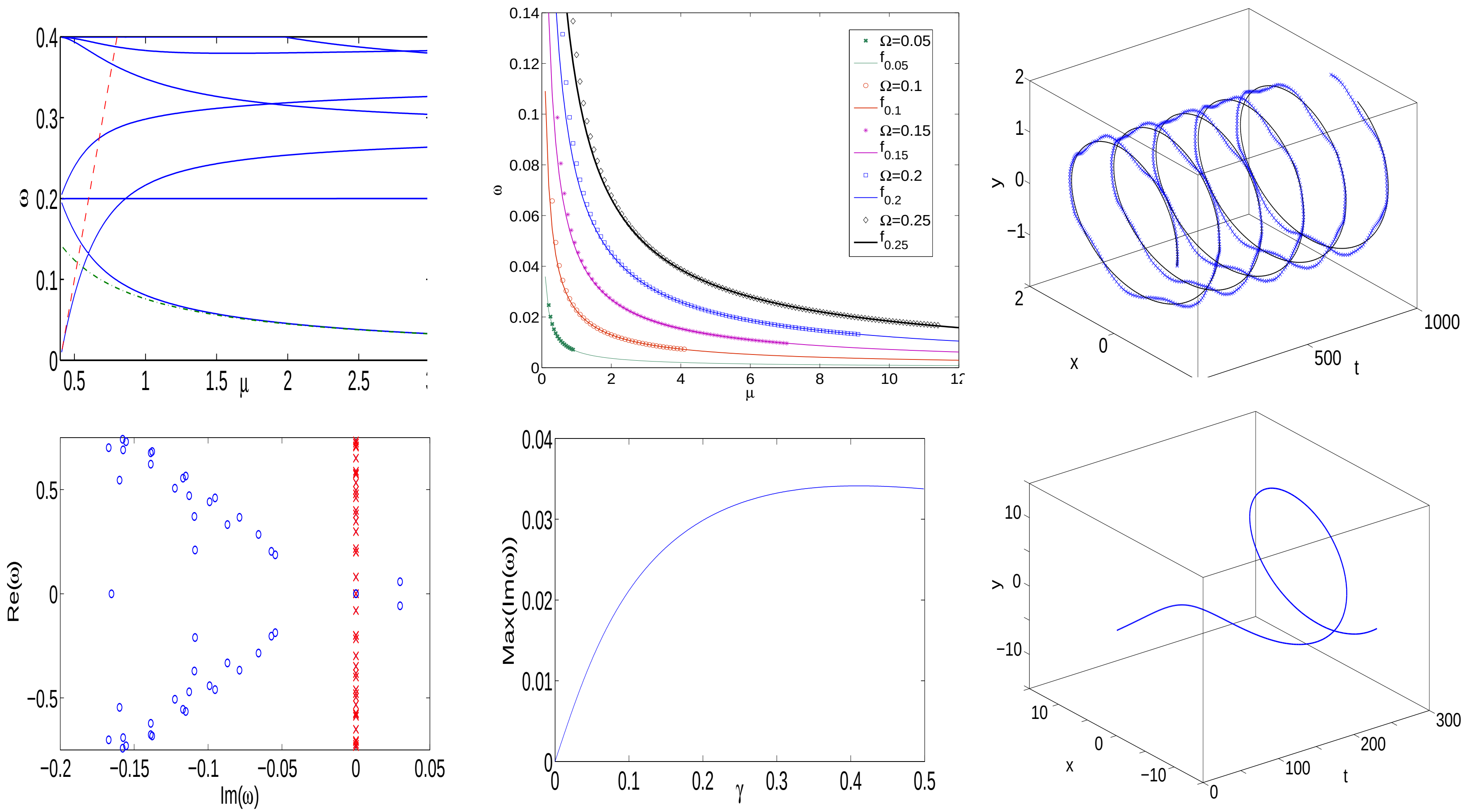
$$\dot{x}_v = \frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) y_v, \tag{55}$$

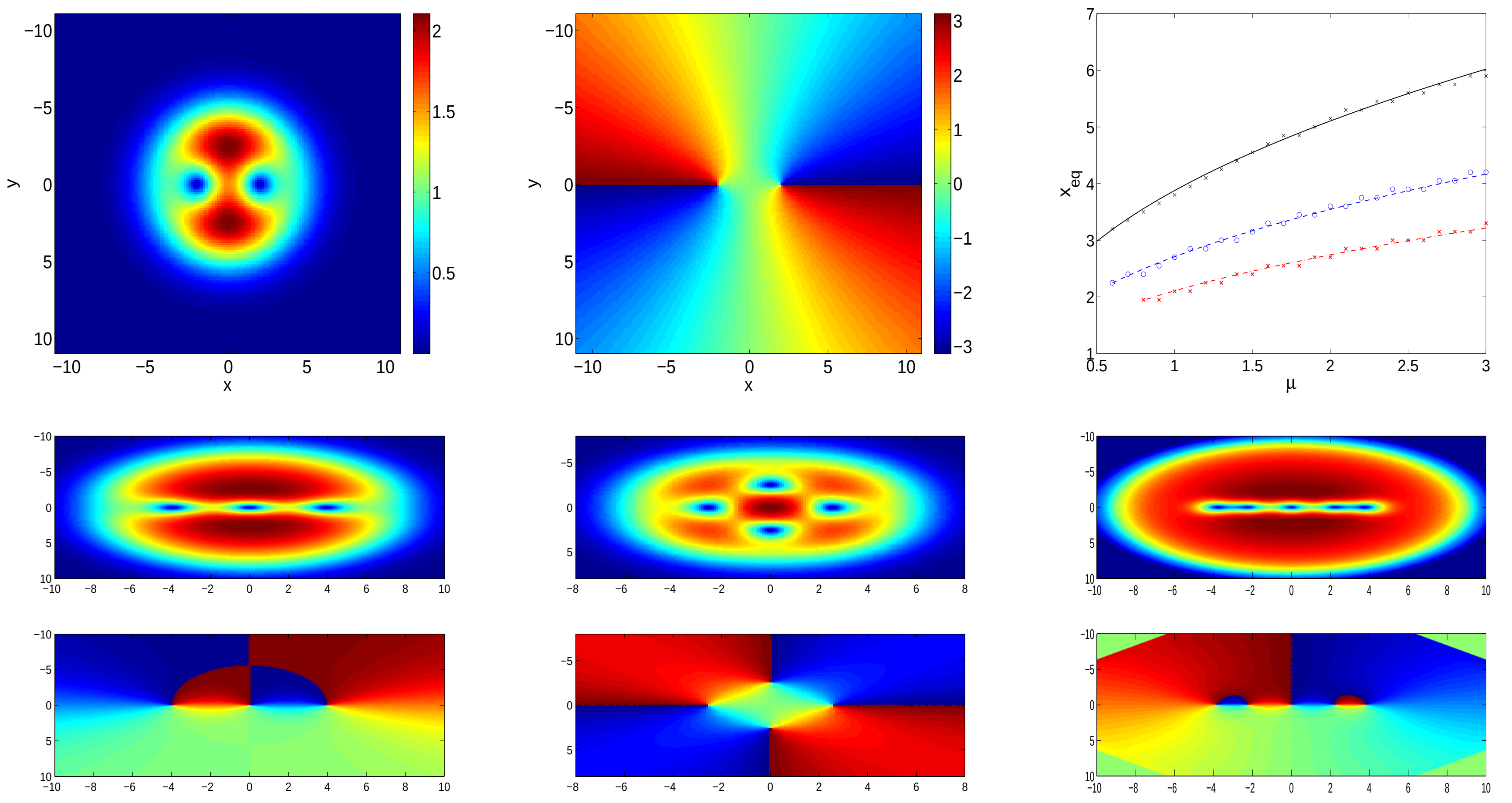
$$\dot{y}_v = -\frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) x_v, \tag{56}$$

- This yields the Precession Frequency:

$$\omega_{\text{prec}} = \frac{\Omega^2}{2\mu} \log(A\frac{\mu}{\Omega}) \tag{57}$$

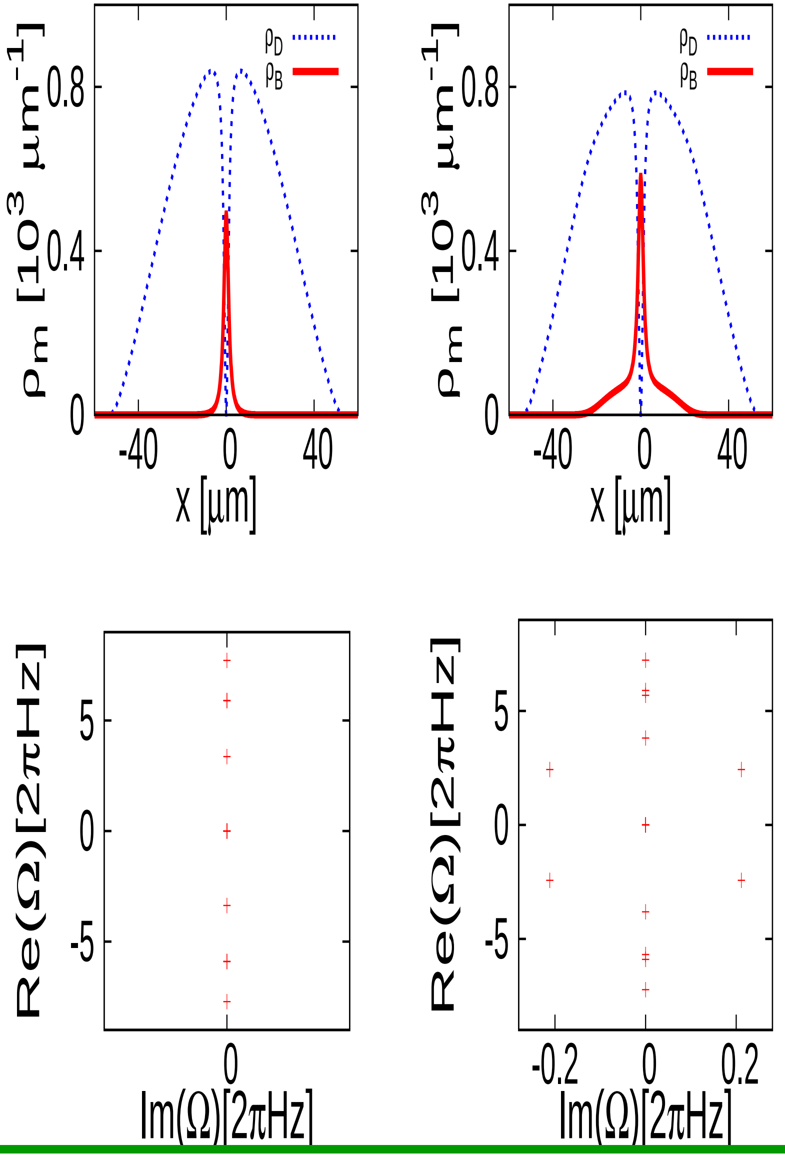
- In the presence of Finite Temperature, this mode becomes Complex giving rise to Dynamical Instability.


ω
μ
Ω=0.05
f0.05
Ω=0.1
f0.1
Ω=0.15
f0.15
Ω=0.2
f0.2
Ω=0.25
f0.25
y
x
t
Re(ω)
Im(ω)
Max(Im(ω))
γ



x
y
x_{eq}
μ

One Example From Each (Dark-Bright Solitons and their Spectra)



One Example From Each (Dark Solitons & Double Wells)

- The presence of an Optical Lattice imposes an Effective Soliton Potential

$$V_{eff}(z) = \frac{1}{2}\omega_{ds}^2 z^2 + \frac{1}{4}V_0 \cos(2kz). \tag{58}$$

- Then, a Bifurcation can arise even for a Single Soliton when $z_0 = \frac{kV_0}{2\omega_{ds}^2}\sin(2kz_0)$. i.e., when $V_0 > V_{0,cr} = \frac{\omega_{ds}^2}{k^2}$.

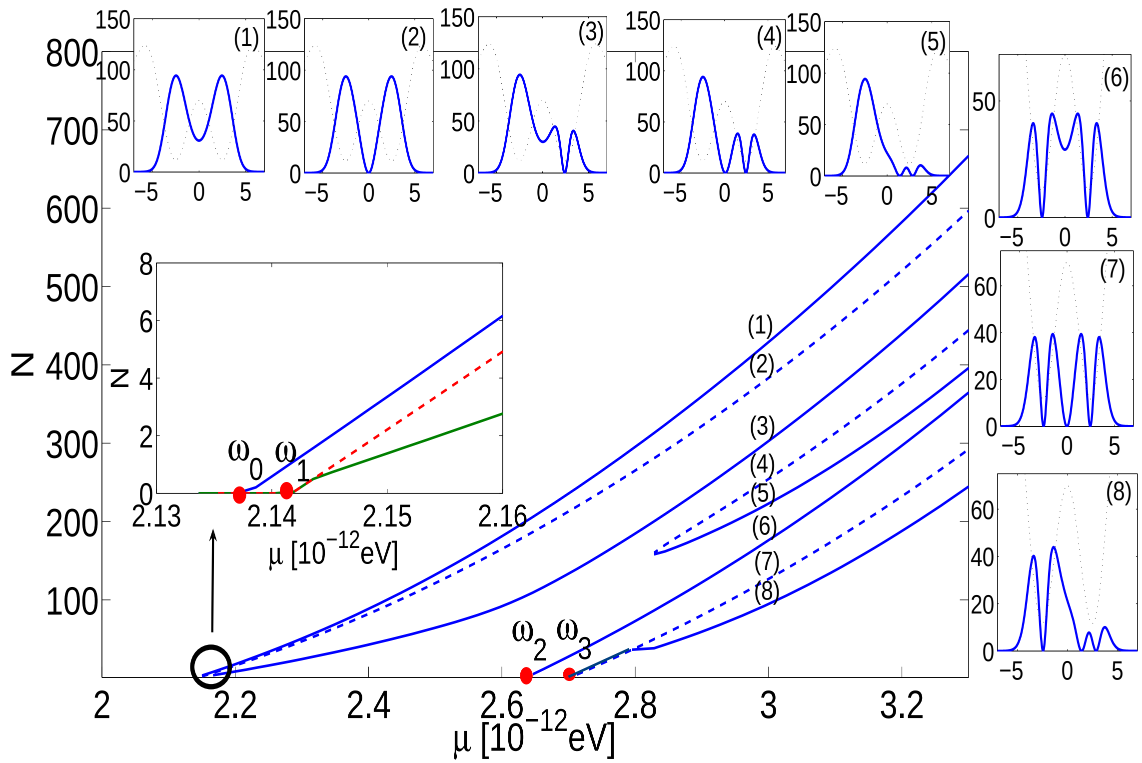
- For Multiple Solitons accounting for the Intersoliton Interactions, we obtain the Equations of Motion

$$\frac{d^2 z_1}{dt^2} = V'_{eff}(z_1) - 2n_0^{\frac{3}{2}} \coth(\sqrt{n_0}(z_2 - z_1))\text{csch}^2(\sqrt{n_0}(z_2 - z_1)), \tag{59}$$

$$\frac{d^2 z_2}{dt^2} = V'_{eff}(z_2) + 2n_0^{\frac{3}{2}} \coth(\sqrt{n_0}(z_2 - z_1))\text{csch}^2(\sqrt{n_0}(z_2 - z_1)). \tag{60}$$

- One can then draw the Bifurcation Structure as a function of both the Chemical Potential μ and the Optical Lattice Strength V_0.

Double Well Potential (Fixed V_0, Vary μ)



Double Well Potential (Fixed μ, Vary V_0)