Spontaneous Stochasticity and Flux-Freezing in Turbulent Plasmas

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Richardson Two-Particle Diffusion

Volcanic ash plume over Kīlauea volcano

Meteorologist and applied mathematician Lewis Fry Richardson proposed in 1926 that particle-pairs advected by turbulence (e.g. a pair of soot particles in a volcanic plume) would have mean-square separation increasing with time as the cube power

$$\langle |\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2 \rangle \sim t^3.$$

This is Richardson's t^3-law.

Scale-Dependent Eddy-Diffusivity

Reference.	K cm.^{2} sec^{-1}	l cm.
K from molecular diffusion of oxygen into nitrogen (Kaye and Laby's 'Physical and Chemical Constants'). For l see preceding discussion.	$1\cdot7 \times 10^{-1}$	5×10^{-2}
K at 9 metres above ground from anemometers at heights of 2, 16 and 32 metres (W. Schmidt, 'Wien. Akad. Sitzb.,' IIa, vol. 126, p. 773 (1917)).	$3\cdot2 \times 10^3$	$1\cdot5 \times 10^3$
K from anemometers at heights of 21 to 305 metres (Åkerblom, F., 'Nova Acta Reg. Soc. Upsaliensis' (1908)).	$1\cdot2 \times 10^5$	$1\cdot4 \times 10^4$
K from pilot balloons at heights between 100 and 800 metres (Taylor, 'Phil. Trans.,' A, vol. 215, p. 21 (1914), also Hesselberg and Sverdrup, 'Leipzig Geophys. Inst.,' Ser. 2, Heft 10 (1915)).	6×10^4	5×10^4
K from tracks of balloons either manned (L. F. Richardson, 'Weather Prediction by Numerical Process,' p. 221) or not manned (Richardson & Proctor, 'Royal Meteorological Society Memoirs,' No. 1).	10^8	2×10^6
Volcano ash, same reference as last	5×10^8	5×10^6
Diffusion due to cyclones regarded as deviations from the mean circulation of the latitude (Defant, 'Geog. Ann.,' H. 3, also (1921), 'Wien. Akad. Wiss. Sitzb.,' IIa, vol. 130, p. 401 (1921)).	10^{11}	10^8

Richardson's table of raw data

Richardson's approach was semi-empirical. By estimating "effective diffusivity" $K = \langle|\Delta\mathbf{x}|^2\rangle/t$ as a function of $\ell = \sqrt{\langle|\Delta\mathbf{x}|^2\rangle}$, he found from data that

$$K(\ell) \sim K_0 \ell^{4/3}.$$

He proposed that the probability density function of the separation vector $\boldsymbol{\ell} = \mathbf{x}_1 - \mathbf{x}_2$ would satisfy a diffusion equation

$$\partial_t P(\boldsymbol{\ell}, t) = \frac{\partial}{\partial \ell_i}\left(K(\ell)\frac{\partial P}{\partial \ell_i}(\boldsymbol{\ell}, t)\right)$$

with scale-dependent 2-particle eddy-diffusivity. This equation predicts at long times that

$$\langle|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^2\rangle \sim t^3,$$

averaging over velocity realizations.

Similarity Solution

Richardson (1926) observed that there is an exact similarity solution of his equation, given by the stretched-exponential PDF

$$P_*(\boldsymbol{\ell}, t) = \frac{A}{(K_0 t)^{9/2}} \exp\left(-\frac{9\ell^{2/3}}{4K_0 t}\right)$$

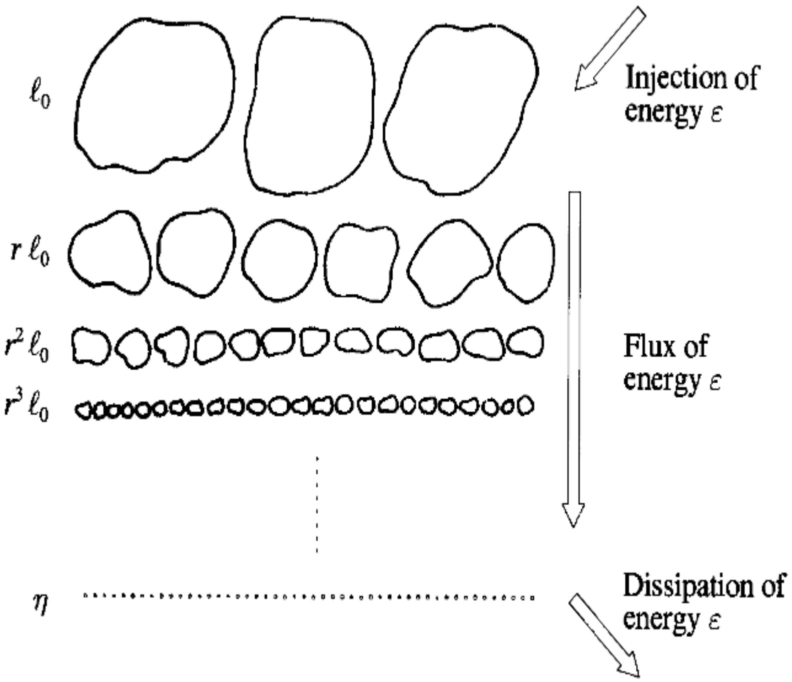
in three space dimensions. All solutions approach this self-similar form asymptotically at long times.

Averaging ℓ^2 with respect to this density yields

$$\langle \ell^2(t) \rangle = \gamma_0 t^3$$

with $\gamma_0 = 1144 K_0^3/81$.

Kolmogorov Cascade Picture



A cartoon of the Kolmogorov cascade

In the Kolmogorov (1941) picture, velocity differences across eddies of size ℓ have magnitude

$$\delta u(\ell) \sim (\varepsilon\ell)^{1/3}.$$

This increases with ℓ, so that larger turbulent eddies have larger velocities.

A pair of particles as they separate thus experience greater relative velocities as they move further apart. The outcome is an explosive separation

$$\langle \ell^2(t) \rangle \sim g_0 \varepsilon t^3,$$

even much faster than ballistic ($\propto t^2$).

The (presumed universal) constant g_0 is now usually called "Richardson's constant".

Advection by Kolmogorov Velocity

A toy calculation: Assume that $\ell(t)$ satisfies

$$\frac{d}{dt}\ell(t) = \delta u(\ell) = (g_0 \varepsilon \ell)^{1/3}.$$

Separation of variables gives the exact solution

$$\ell(t) = \left[\ell_0^{2/3} + (g_0 \varepsilon)^{1/3}(t - t_0)\right]^{3/2}.$$

For $t - t_0 \gg \ell_0^{2/3}/(g_0\varepsilon)^{1/3} \equiv T_0$

$$\ell^2(t) \sim g_0 \varepsilon t^3.$$

Fate of Particles Initially at the Same Point?

An odd feature of the previous result is that, if $\ell_0 = 0$, then

$$\ell^2(t) = g_0 \varepsilon (t - t_0)^3 > 0.$$

Two particles started at the *same* point at time t_0 separate to a finite distance at any time $t > t_0$!

The same oddity may be seen in Richardson's similarity solution, which satisfies at initial time $t_0 = 0$

$$P_*(\boldsymbol{\ell}, 0) = \delta^3(\boldsymbol{\ell}).$$

All particles start with separation $\ell(0) = 0$. However, $P_*(\boldsymbol{\ell}, t)$ is a smooth density for $t > 0$, so that $\ell(t) > 0$ with probability one.

Breakdown of Laplacian Determinism

According to Richardson's results, Lagrangian fluid particles that are advected by the fluid velocity $\mathbf{u}(\mathbf{x},t)$ starting at $\mathbf{x}_0$

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t),t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

have the property that there is more than one solution. Doesn't this violate the theorem on uniqueness of solutions of initial-value problems for ODE's? No!

Loophole: The theorem requires that $\mathbf{u}(\mathbf{x},t)$ be $\mathbf{x}-$differentiable. A turbulent velocity field in a Kolmogorov inertial range is only Hölder continuous

$$|\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t)| \le C|\mathbf{x}_1 - \mathbf{x}_2|^h$$

with exponent $h \doteq 1/3$.

Kraichnan White-Noise Advection Model

All the previous facts were noted in a seminal paper of Bernard, Gawędzki, and Kupiainen (1998). They studied a soluble model of advection by a Gaussian random velocity field with zero mean and covariance

$$\langle u_i^\nu(\mathbf{x},t)u_j^\nu(\mathbf{x}',t')\rangle = [D_0^\nu\delta_{ij} - S_{ij}^\nu(\mathbf{x}-\mathbf{x}')]\delta(t-t')$$

temporal white-noise and spatially rough for $r_\nu \ll r \ll L$,

$$S_{ij}^\nu(\mathbf{r}) = D_1[(1+h)\delta_{ij} - h\widehat{r}_i\widehat{r}_j]r^{2h}$$

with $0 < h < 1$, but smooth for $r < r_\nu$. The velocity realizations $\mathbf{u}^\nu(\mathbf{x},t)$ are incompressible and only Hölder continuous with exponent h for $\nu \equiv D_1 r_\nu^{2h} \to 0$.

Richardson's 2-particle diffusion equation holds exactly within this model

$$\partial_t P(\mathbf{r},t) = \frac{\partial}{\partial r_i}\left(S_{ij}^\nu(\mathbf{r})\frac{\partial P}{\partial r_j}(\mathbf{r},t)\right)$$

with $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. Note Richardson's original equation corresponds to $h = 2/3$ (a peculiarity of the white-noise approximation).

Random Advection Problem

Bernard et al. (1998) study the problem of stochastic particle advection,

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{u}^{\nu}(\mathbf{x}(t),t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

perturbed by a Gaussian white-noise $\boldsymbol{\eta}(t)$ multiplied by $\sqrt{2\kappa}$.

The transition probability for a single particle in a fixed (non-random) velocity realization $\mathbf{u}$ can be written as Feynman-type path-integral:

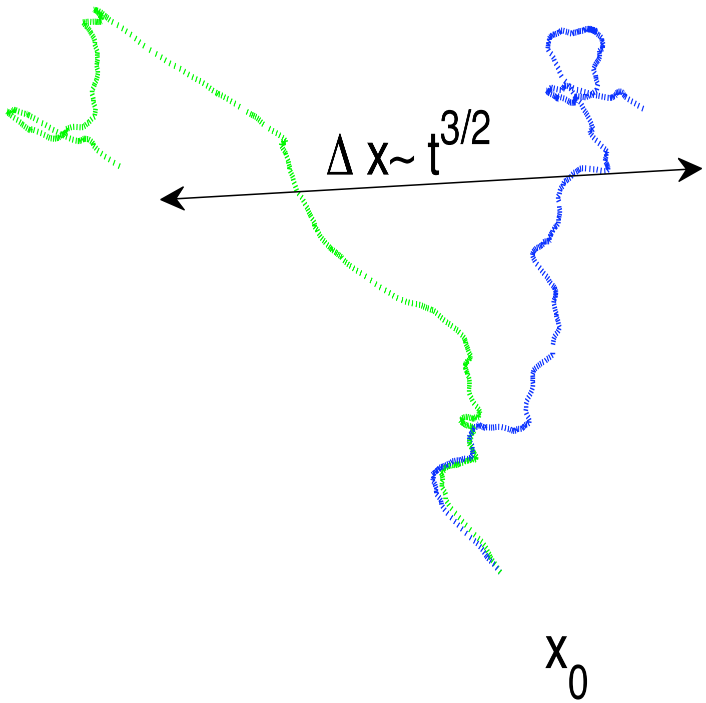
$$P_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x},t|\mathbf{x}_0,t_0) = \int_{\mathbf{x}(t_0)=\mathbf{x}_0}^{\mathbf{x}(t)=\mathbf{x}} \mathcal{D}\mathbf{x}\ \exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau\,|\dot{\mathbf{x}}(\tau) - \mathbf{u}^{\nu}(\mathbf{x}(\tau),\tau)|^2\right)$$

In the limit $\nu,\kappa \to 0$ (or infinite Reynolds-number $Re = u_{rms}L/\nu$) with $Pr = \nu/\kappa$ fixed one would naively expect collapse to a delta-function,

$$P_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x},t|\mathbf{x}_0,t_0) \to \delta^3(\mathbf{x} - \mathbf{x}(t)),$$

with $\mathbf{x}(t)$ the unique solution of $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x},t),\ \mathbf{x}(t_0) = \mathbf{x}_0$.

Spontaneous Stochasticity



The distribution does not collapse! At least in the Kraichnan model there is a nontrivial limiting distribution $P_{\mathbf{u}}(\mathbf{x},t|\mathbf{x}_0,t_0)$ over an infinite family of solutions to the initial-value problem $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x},t),\ \ \mathbf{x}(t_0) = \mathbf{x}_0$. (Le Jan & Raimond, 2002, 2004)

There is an obvious analogy with *spontaneous symmetry-breaking*, e.g. a non-vanishing mean-magnetization in a ferromagnet even in the limit of zero external magnetic field.

More Chaotic Than Chaos!

Compare with what happens for a smooth velocity field ($h = 1$):

$$\frac{d}{dt}\ell(t) = \delta u(\ell) \doteq A\ell.$$

Separation of variables now gives the exact solution

$$\ell(t) = \ell_0 e^{A(t-t_0)}.$$

The trajectories never "forget" their initial separations: $\ell(t) \to 0$ as $\ell_0 \to 0$ for any $t > t_0$.

For smooth dynamical systems there is at most exponential deviation of trajectories. The exponential growth rate

$$\lambda = \lim_{t\to\infty} \lim_{\ell_0\to 0} \frac{1}{t-t_0} \left(\frac{\ell(t)}{\ell_0}\right)$$

is the Lyapunov exponent and $\lambda > 0$ is the signature of chaos. Any imprecision in the initial data is exponentially magnified, leading to loss of predictability at long enough times.

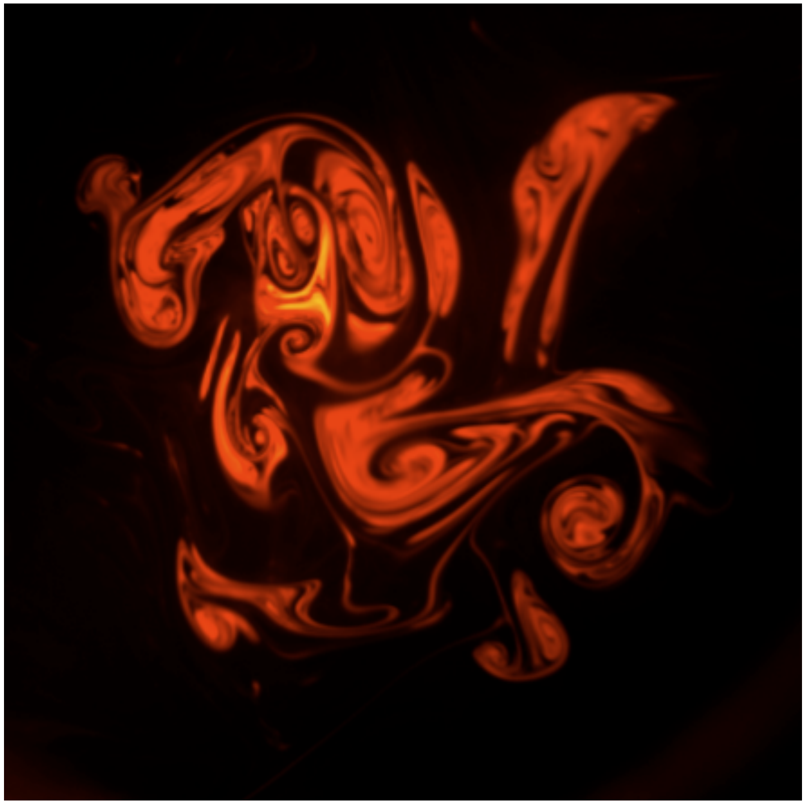
Spontaneous stochasticity corresponds instead to $\lambda = +\infty$. The solution is unpredictable for *all* future times, even with infinitely precise knowledge of the initial conditions!

Passive Scalar Advection

The scalar advection-diffusion equation

$$\partial_t\theta + (\mathbf{u}^\nu \cdot \boldsymbol{\nabla})\theta = \kappa\triangle\theta,$$

describes the evolution of dye and other passive tracers in a turbulent flow:



The exact solution is given by the Feynman-Kac formula

$$\theta(\mathbf{x},t) = \int d^3x_0\, \theta(\mathbf{x}_0,t_0) P_{\mathbf{u}}^{\nu,\kappa}(\mathbf{x}_0,t_0|\mathbf{x},t)$$

$$= \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\ \theta(\mathbf{a}(t_0),t_0)$$

$$\exp\left(-\frac{1}{4\kappa}\int_{t_0}^{t} d\tau\, |\dot{\mathbf{a}}(\tau) - \mathbf{u}^\nu(\mathbf{a}(\tau),\tau)|^2\right)$$

for $t_0 < t$. Now the stochastic equation

$$\dot{\mathbf{a}}(\tau) = \mathbf{u}^\nu(\mathbf{a}(\tau),\tau) + \sqrt{2\kappa}\,\boldsymbol{\eta}(\tau)$$

is solved backward in time from t to t_0, with final condition $\mathbf{a}(t) = \mathbf{x}$.

Dissipative Anomaly

Note that

$$\frac{d}{dt}\int d^3x\ \theta^2(\mathbf{x},t) = -2\kappa\int d^3x\ |\boldsymbol{\nabla}\theta(\mathbf{x},t)|^2,$$

so that, naively, the integral is conserved for $\kappa = 0$.

However, in the infinite Reynolds-number, fixed Prandtl-number limit ($\nu,\kappa \to 0$)

$$\theta(\mathbf{x},t) = \int d^3x_0\,\theta(\mathbf{x}_0,t_0)P_{\mathbf{u}}(\mathbf{x}_0,t_0|\mathbf{x},t).$$

Heuristically, molecular diffusion is replaced by turbulent diffusion. In particular, the scalar intensity is still dissipated

$$\int d^3x\ \theta^2(\mathbf{x},t) < \int d^3x\ \theta^2(\mathbf{x},t_0), \quad t > t_0$$

even as $\nu,\kappa \to 0$!

This is the scalar analogue of a dissipative anomaly of Onsager (1949) for fluid turbulence.

The Lagrangian mechanism is "spontaneous stochasticity."

Laboratory Experiment

M. Bourgoin et al. Science **311** 835–38 (2006)

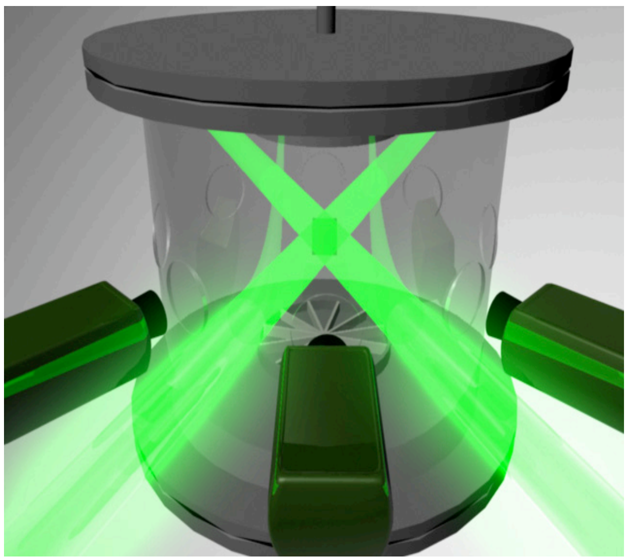
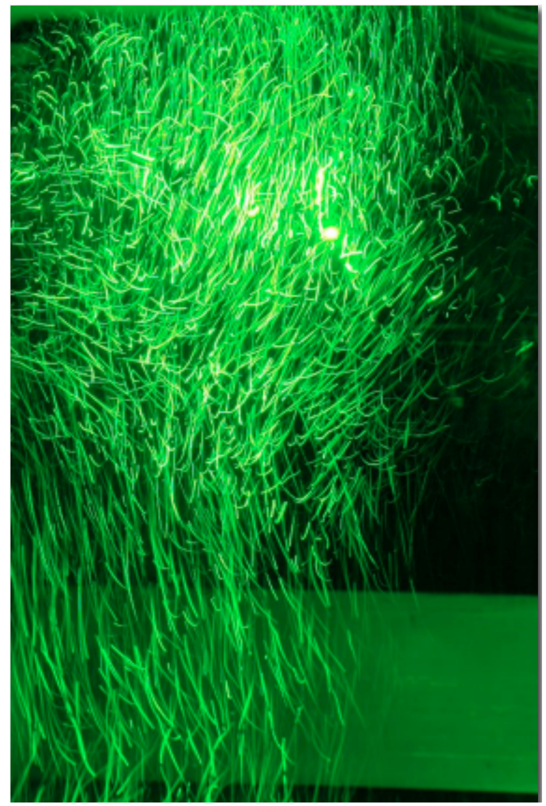


Figure 1. Sketch of the experimental setup. The three cameras have an angular separation of approximately 45° and are arranged in the forward scattering direction from both lasers.



Experimental Results

Reynolds numbers up to about 5×10^4

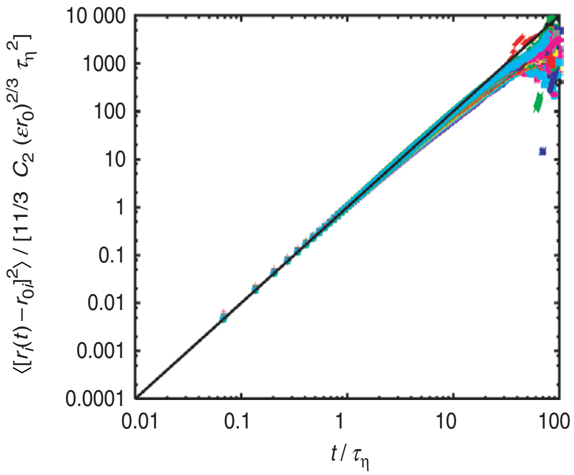


Figure 3. Scale collapse of the mean-square particle separation at $R_\lambda = 815$. The relative dispersion $\langle|\mathbf{r}(t) - \mathbf{r}_0|^2\rangle$ scaled by $(11/3)C_2(\epsilon r_0)^{2/3}$ is plotted for 50 different bins of initial separations, ranging from 0–1 mm ($\approx$0–43η) to 49–50 mm ($\approx$2107–2150η). The solid line is a pure t^2 power law, and is not a fit. The data collapse on to the t^2 law almost perfectly [21].

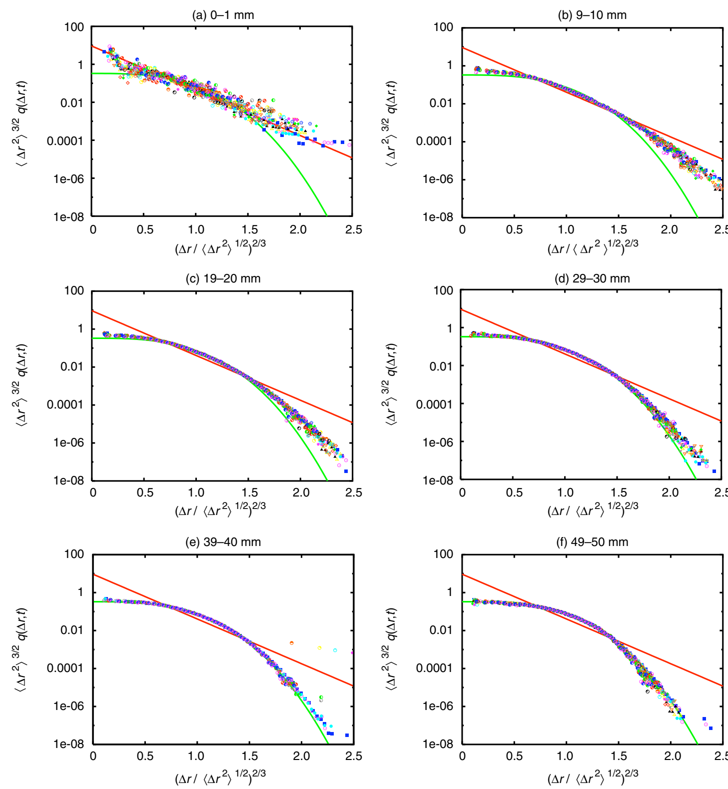


Figure 10. The distance neighbour function for different initial separations at $R_\lambda = 815$. The red straight line is Richardson's predicted PDF, while the green curved line is Batchelor's. The symbols show the experimental measurements. Each plot shows a different initial separation; for each initial separation, PDFs from 20 times ranging from τ_η to $20\tau_\eta$ are shown.

Modified Richardson Scaling

More success in looking for the modified Richardson scaling law:

$$\langle r^{2/3}(t)\rangle - r_0^{2/3} \sim C_R \varepsilon^{1/3} t.$$

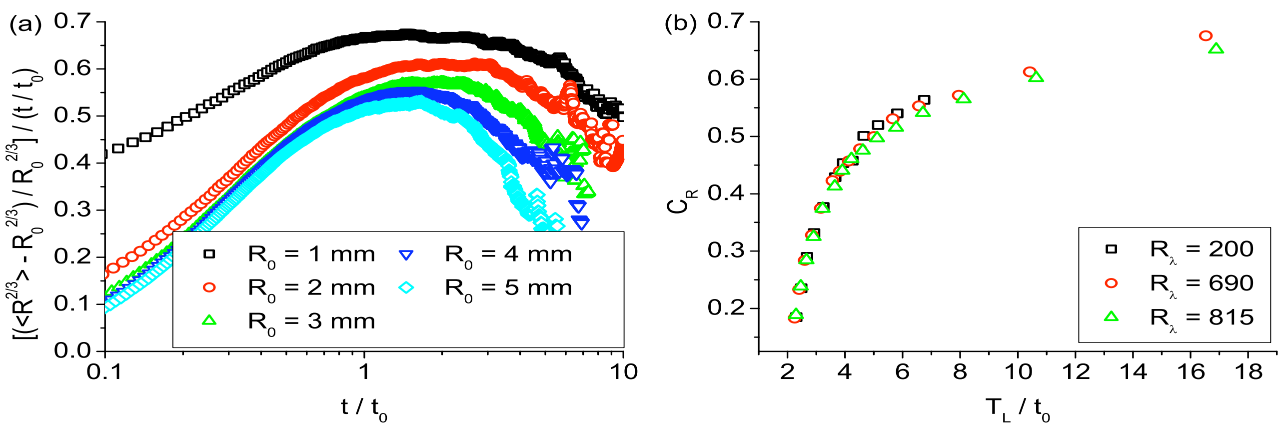


Figure 1. (a) The separation of two particles in time, compared with a modified Richardson–Obukhov law, for different initial separations, ranging from 1 to 5 mm. The Reynolds number is fixed at $R_\lambda = 690$, with a Kolmogorov scale of $\eta = 30\,\mu$m. We observe similar behavior at other Reynolds numbers. (b) The change of C_{R} with T_{L}/t_0, shown for three Reynolds numbers.

Note that $t_0 \equiv (r_0^2/\varepsilon)^{1/3}$.

Göttingen Turbulence Tunnel

Apparatus	P (bar)	ν (m^2/s)	u' (m/s)	ϵ (m^2/s^3)	ℓ (m)	λ (μm)	η (μm)	τ_η (ms)	R_λ
SF_6 tunnel	15	1.5×10^{-7}	1.0	1.2	0.45	1400	7.3	0.36	9600
air tunnel	1	1.5×10^{-5}	1.2	3.9	0.4	9100	172	2.0	730
SF_6 tank	15	1.5×10^{-7}	1.0	5.5	0.094	648	5.0	0.17	4360
water tank	1	8×10^{-7}	2.2	59	0.094	1000	9.7	0.12	2800

Numerical Simulations

L. Biferale et al. Phys. Fluids **17** 115101 (2005) employed numerical simulations of the Navier-Stokes fluid equations on a 1024^3 grid at Reynolds numbers up to 6066

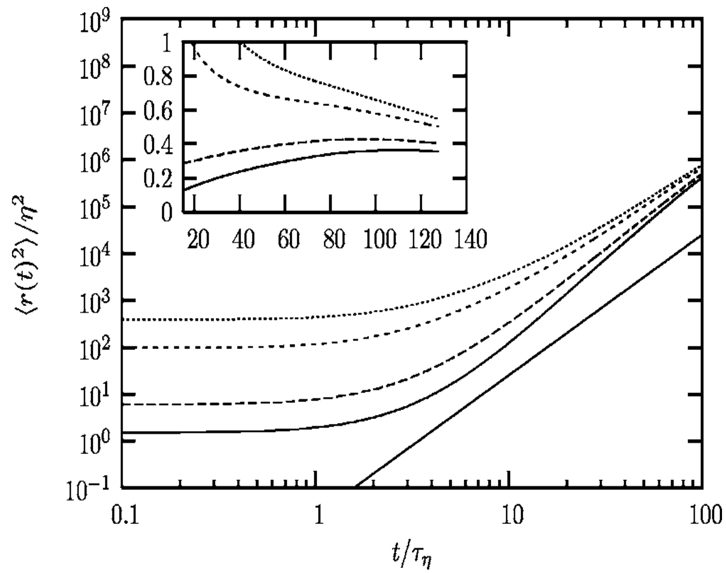


FIG. 1. The evolution of $\langle r(t)^2\rangle/\eta^2$ vs t/τ_η for the initial separations $r_0=1.2\eta$, $r_0=2.5\eta$, $r_0=9.8\eta$, and $r_0=19.6\eta$. The straight line is proportional to t^3. Inset: $\langle r(t)^2\rangle/\varepsilon t^3$ for the same four initial separations starting from $t/\tau_\eta \sim 15$.

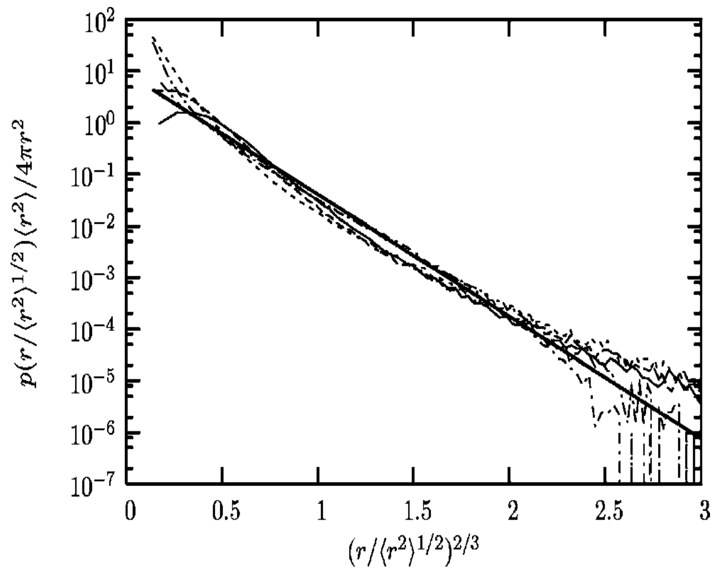


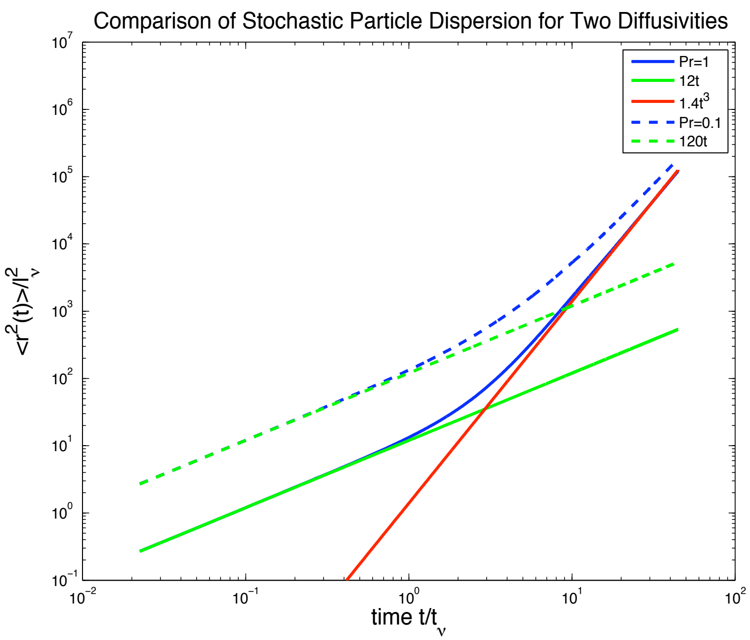
FIG. 2. Comparison of the Richardson PDF with the DNS data. The curves refer to data for $r_0=1.2\eta$ at $t=5.2\tau_\eta$ (solid line), $t=7\tau_\eta$ (long dashed line), $t=14\tau_\eta$ (short dashed line), $t=42\tau_\eta$ (dotted line), and $t=70\tau_\eta$ (dot-dashed line). The thick solid line is the Richardson PDF (2).

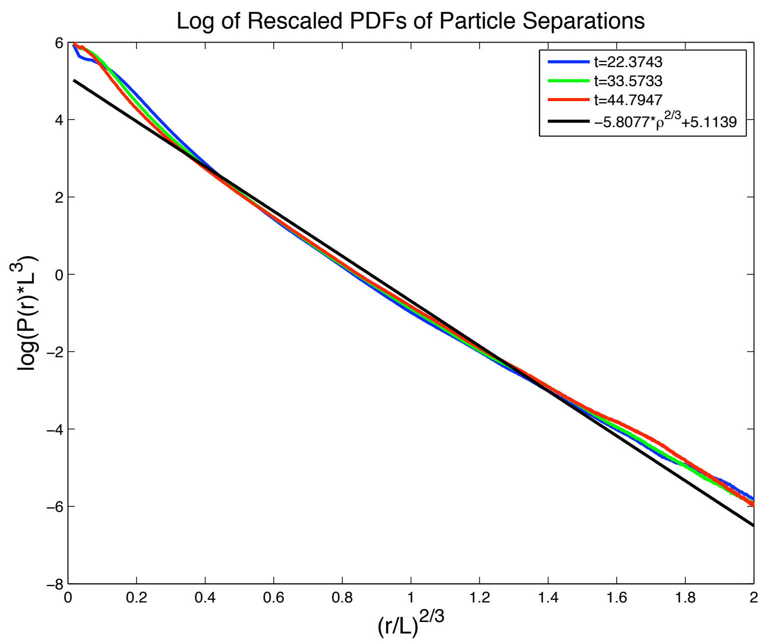
My Numerical Results!

I have performed a study using turbulence data from numerical simulations on a 1024^3 grid at Reynolds number 5065, available on-line in the public database at Johns Hopkins:

`http://turbulence.pha.jhu.edu`

I track 1024 particles (5×10^5 pairs) that all satisfy $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$. Below $\kappa = \nu$ and 10ν. The results are averaged over 256 initial points $\mathbf{x}_0$:





Plasma Magnetohydrodynamics

Hannes Alfvén, physics Nobel laureate (1970) — "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics"

Sufficiently collisional plasmas obey the magnetohydrodynamics equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{4\pi\rho}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\triangle\mathbf{u}$$

$$\partial_t \mathbf{B} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} + \lambda\triangle\mathbf{B}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0,$$

for an incompressible plasma. $\nu = \mu/\rho$ is the kinematic viscosity and $\lambda = \eta c/4\pi$ is the magnetic diffusivity for the (Spitzer) resistivity η.

The first MHD equation is Newton's 2nd law and the second MHD equation—called the induction equation—is a combination of Faraday's law and Ohm's law.

Magnetic Flux-Freezing

Ideal MHD ($\lambda = 0$) predicts that magnetic flux is "frozen-in" to the plasma (Alfvén, 1942): magnetic field lines are advected by the plasma fluid velocity.

This result is usually formulated as the *Alfvén Theorem* on conservation of magnetic flux through a material (Lagrangian) loop:

$$\int_S \mathbf{B}_0(\mathbf{a})\cdot d\mathbf{S}(\mathbf{a}) = \int_{\mathbf{x}_t(S)} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S}(\mathbf{r}).$$

It is widely assumed in astrophysics and plasma physics that magnetic flux is "nearly frozen-in" for high conductivity (or high magnetic Reynolds numbers).

"The most important property of an ideal plasma is flux freezing." —R. M. Kulsrud, *Plasma Physics for Astrophysics* (2005)

The Magnetic Reconnection Problem

It is usually argued that magnetic reconnection cannot be explained by resistive MHD, because the violations of flux-freezing are too slow at high conductivity:

"Flux freezing is a very strong constraint on the behavior of magnetic fields in astrophysics. As we show in chapter 3, this implies that lines do not break and their topology is preserved. The condition for flux freezing can be formulated as follows: In a time t, a line of force can slip through the plasma a distance

$$\ell = \sqrt{\frac{\eta c t}{4\pi}} \tag{1}$$

If this distance ℓ is small compared to δ, the scale of interest, then flux freezing holds to a good degree of approximation."—R. M. Kulsrud (2005), Ch.13, Magnetic Reconnection

Most attempts to explain *fast magnetic reconnection* appeal to microscopic plasma mechanisms besides Spitzer resistivity which can decouple plasma particles and field lines over larger distances, e.g. anomalous resistivity, Hall MHD effect, etc.

Breakdown of Flux-Freezing in Turbulent Plasmas

We argue instead that flux-freezing is *strongly broken* in MHD even at vanishingly small resistivities, by the turbulent effect of spontaneous stochasticity. We saw earlier that

$$\ell^2 \propto \lambda t$$

holds only for very short times, $t \ll \sqrt{\lambda/\varepsilon}$, whereas for $t \gg \sqrt{\lambda/\varepsilon}$

$$\ell^2 \propto \varepsilon t^3$$

independent of λ! Note that most high-conductivity plasmas are automatically turbulent.

In fact, there is a paradox of flux-freezing: It is usually argued that flux-freezing should hold better and better as $\lambda \to 0$. However, if $\nu \to 0$ simultaneously and the plasma becomes turbulent, then Lagrangian trajectories are no longer unique. Which fluid trajectory shall a magnetic field line follow if there are infinitely many such trajectories???

Stochastic Flux Freezing for Non-Ideal Plasmas

The exact solution of the induction equation

$$\partial_t \mathbf{B} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u} + \lambda\triangle\mathbf{B}, \qquad \mathbf{B}(t_0) = \mathbf{B}_0$$

is given by the "sum-over-histories" formula

$$\mathbf{B}(\mathbf{x},t) = \int_{\mathbf{a}(t)=\mathbf{x}} \mathcal{D}\mathbf{a}\ \mathbf{B}_0(\mathbf{a}(t_0))\cdot\mathbf{J}(\mathbf{a},t,t_0)\,\exp\left(-\frac{1}{4\lambda}\int_{t_0}^{t} d\tau\,|\dot{\mathbf{a}}(\tau) - \mathbf{u}^{\nu}(\mathbf{a}(\tau),\tau)|^2\right)$$

where the matrix $\mathbf{J}$ satisfies the ODE along the stochastic trajectory $\mathbf{a}(\tau)$

$$\frac{d}{d\tau}\mathbf{J}(\mathbf{a},\tau,t_0) = \mathbf{J}(\mathbf{a},\tau,t_0)\boldsymbol{\nabla}_x\mathbf{u}(\mathbf{a}(\tau),\tau), \quad \mathbf{J}(\mathbf{a},t_0,t_0) = \mathbf{I}.$$

See G. L. Eyink, "Stochastic line motion and stochastic flux conservation for nonideal hydromagnetic models," J. Math. Phys. **50** 083102 (2009)

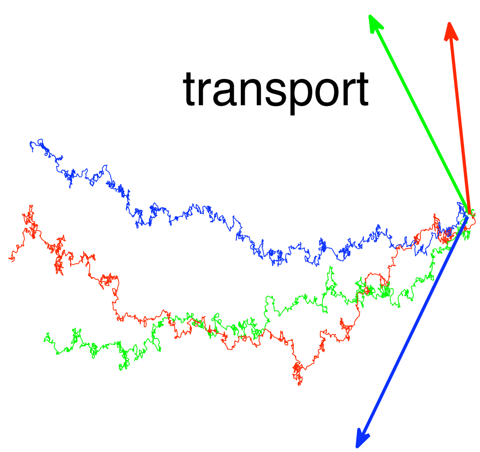
Note that the final condition $\mathbf{a}(t) = \mathbf{x}$ on the path-integral trajectories implies that they correspond to solutions of the stochastic equation

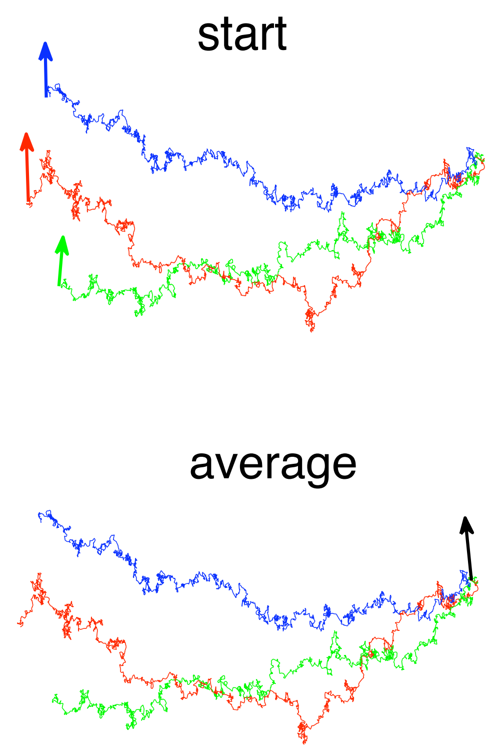
$$\dot{\mathbf{a}}(\tau) = \mathbf{u}(\mathbf{a},\tau) + \sqrt{2\lambda}\,\boldsymbol{\eta}(\tau)$$

integrated backward in time from t to t_0.

Illustration of the Sum-Over-Histories

Three sample trajectories are shown (for a real turbulent velocity field), illustrating how starting magnetic field vectors at the initial time are transported along the stochastic trajectories and then averaged to obtain the resultant magnetic at the final time.



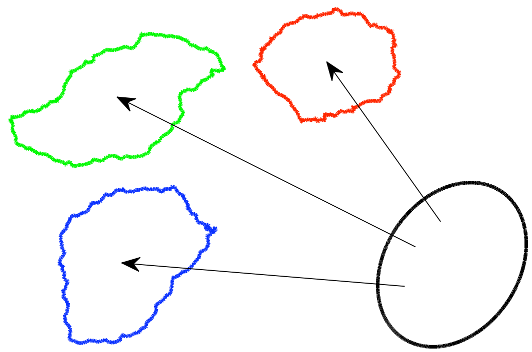


Stochastic Alfvén Theorem

Flux-freezing holds *on average*:

$$\int_S \mathbf{B}(\mathbf{r},t')\cdot d\mathbf{S}(\mathbf{r}) = \left\langle \int_{\mathbf{a}_{t',t}(S)} \mathbf{B}(\mathbf{a},t)\cdot d\mathbf{S}(\mathbf{a}) \right\rangle .$$

Here $\langle\cdot\rangle$ is average over the ensemble of random loops at earlier time:



Note $\mathbf{a}_{t',t} = \mathbf{x}_{t',t}^{-1}$ are "back-to-label maps" for the stochastic forward flows.

The above result holds for all smooth surfaces S and for any pair of times $t_0 \leq t < t' \leq t_f$ if and only if the magnetic field $\mathbf{B}(\mathbf{r},t)$ (for *any* velocity $\mathbf{u}$) satisfies the induction equation $\partial_t\mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) + \lambda\triangle\mathbf{B}$ (Eyink, 2009a).

The Kazantsev-Kraichnan Dynamo Model

Kazantsev (1968) studied the kinematic dynamo model

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{u}^\nu \times \mathbf{B}) + \lambda \triangle \mathbf{B},$$

with Gaussian random velocity field with zero mean and covariance

$$\langle u_i^\nu(\mathbf{x},t) u_j^\nu(\mathbf{x}',t')\rangle = [D_0^\nu \delta_{ij} - S_{ij}^\nu(\mathbf{x}-\mathbf{x}')]\delta(t-t')$$

temporal white-noise and spatially rough for $r_\nu \ll r \ll L$,

$$S_{ij}^\nu(\mathbf{r}) = D_1[(1+h)\delta_{ij} - h\widehat{r}_i\widehat{r}_j]r^{2h}$$

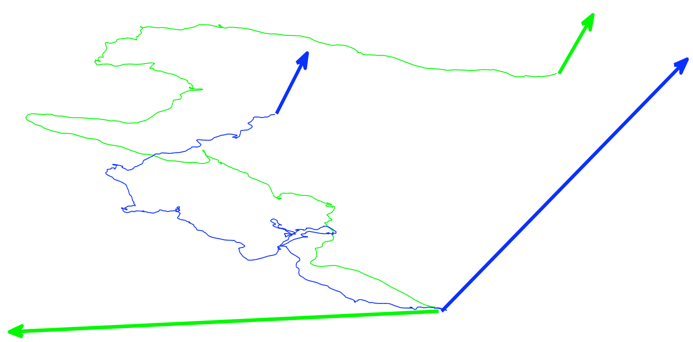
with $0 < h < 1$, but smooth for $r < r_\nu$.

Kazantsev (1968) found for $Pr = \nu/\lambda \ll 1$ that there is a critical roughness exponent h_c, with kinematic dynamo for $h > h_c$ but not for $h < h_c$.

Why? Stretching by velocity-gradients is *much larger* for $h < h_c$!

Explanation of the Dynamo Transition

Eyink & Neto (2009) show that dynamo effect requires sufficient angular correlation of independent pairs of line-vectors advected by the same velocity realization that arrive simultaneously at the same space point:

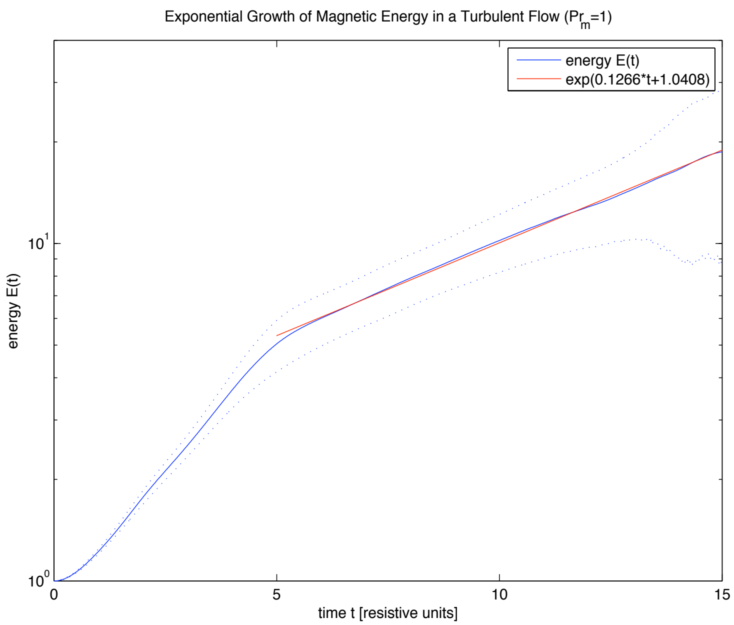


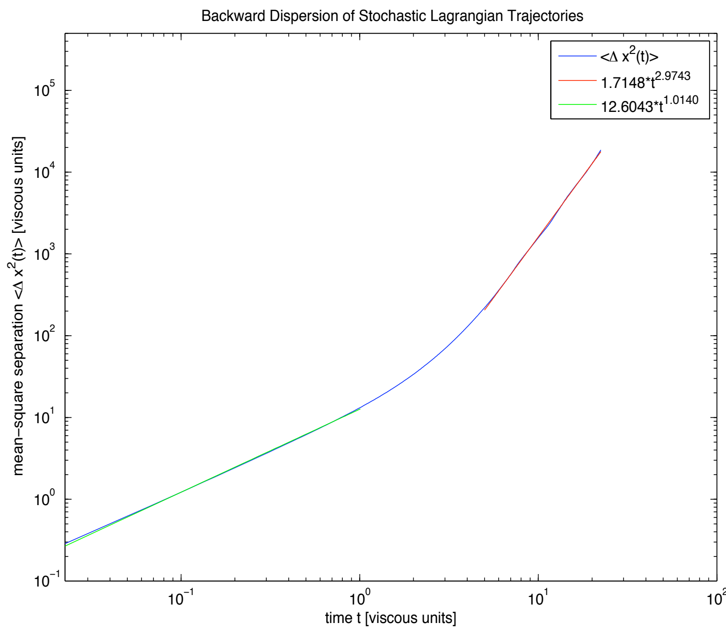
The magnetic vectors arrive from a large volume $\propto R^3(t)$ with $R^2(t) \sim t^{1/(1-h)}$ (analogue of Richardson diffusion).

Many further exact results are given in Eyink & Neto (2009), such as the decay rate of magnetic energy and the properties of "magnetic induction" in the failed dynamo regime for $h < h_c$.

Small-Scale Kinematic Dynamo in Hydrodynamic Turbulence

The stochastic Lundquist formula may be exploited numerically to study dynamo effect in hydrodynamic turbulence by Lagrangian tracking of fluid particles. We show results using 1024^3 hydro DNS data from the JHU Turbulence Database: `http://turbulence.pha.jhu.edu`





The asymptotic exponential growth range and Richardson t^3-range begin at the same time!

Magnetic Reconnection and 2-Particle Dispersion in MHD Turbulence

Unlike in the kinematic dynamo regime, the effects of the Lorentz force are significant in nonlinear MHD turbulence. The properties of 2-particle dispersion depend upon the scaling laws of MHD turbulence.

Assuming the Goldreich-Sridhar (1995) scaling, $\delta u(r_\perp) \sim (\varepsilon r_\perp)^{1/3}$, one obtains the Richardson-type law

$$\ell_\perp^2 \sim \varepsilon t^3$$

for the transverse slippage of magnetic field lines in MHD turbulence.

Two-particle dispersion in MHD turbulence is still poorly understood. E.g.

A. Busse et al., "Statistics of passive tracers in three-dimensional magnetohydrodynamic turbulence," Phys. Plasmas **14** 122303 (2007)

find that particle separation distance $\ell_\|$ parallel to the local magnetic field direction grows faster than separation $\ell_\perp$ perpendicular to the field. Why?

Two Stochastic Invariants at Unit Prandtl Number

(Eyink, 2009a) Divergence-free fields $\mathbf{u}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)$ satisfy the non-ideal, incompressible MHD equations (with $\mathbf{J}=\boldsymbol{\nabla}\times\mathbf{B}$)

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=\mathbf{J}\times\mathbf{B}-\boldsymbol{\nabla}p+\nu\triangle\mathbf{u}, \qquad \partial_t\mathbf{B}=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})+\lambda\triangle\mathbf{B}$$

over the time-interval $[t_0,t_f]$, for unit magnetic Prandtl number $\nu/\lambda=1$, iff for all closed, rectifiable loops C and for any pair of times $t_0\leq t'<t\leq t_f$ (with $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$)

$$\oint_C \mathbf{A}(\mathbf{r},t)\cdot d\mathbf{r}=\left\langle \oint_{\mathbf{a}_{t,t'}(C)} \mathbf{A}(\mathbf{a},t')\cdot d\mathbf{a}\right\rangle \quad \text{(Alfvén Theorem)}$$

$$\oint_C \mathbf{u}(\mathbf{r},t)\cdot d\mathbf{r}=\left\langle \oint_{\mathbf{a}_{t,t'}(C)} [\mathbf{u}(\mathbf{a},t')+\mathbf{B}(\mathbf{a},t')\times\mathbf{R}(\mathbf{a},t,t')]\cdot d\mathbf{a}\right\rangle \quad \text{(Kelvin Theorem)}$$

Here $\mathbf{a}_{t,t'}(\mathbf{x})$ are "back-to-label maps" for stochastic forward flows $\mathbf{x}_{t,t'}(\mathbf{a})$ solving

$$d\mathbf{x}_{t,t'}(\mathbf{a})=\mathbf{u}(\mathbf{x}_{t,t'}(\mathbf{a}),t)dt+\sqrt{2\nu}\,d(t),\ t>t', \qquad \mathbf{x}_{t',t'}(\mathbf{a})=\mathbf{a},$$

and $\mathbf{R}(\mathbf{a},t,t')$ is the *Lagrangian-history charge density* (charge per unit area) satisfying

$$\partial_t\mathbf{R}(\mathbf{a},t,t')=-\mathbf{J}(\mathbf{x}_{t,t'}(\mathbf{a}),t)(\boldsymbol{\nabla}_a\mathbf{x}_{t,t'}(\mathbf{a}))^{-1},\ t>t', \qquad \mathbf{R}(\mathbf{a},t',t')=\mathbf{0}.$$

Cf. Bekenstein & Oron (2000), Kuznetsov & Ruban (2000) for ideal MHD.

Further Extensions

* Hall MHD, two-fluid plasma equations (Eyink, 2009a)

* compressible flow, anisotropic transport (Eyink, in preparation)

These results all follow from a stochastic least-action principle for non-ideal (resistive & viscous) magnetofluids, which extends the classical Maupertuis principle to Hamiltonian fluids plus Laplacian-dissipation.

The stochastic Lagrangian conservation laws are the consequence of infinite-dimensional particle-relabelling symmetry groups of the stochastic action functional, by a Noether Theorem argument.

See Eyink (2009b) for a neutral viscous fluid.

Conclusion:

Flux-Freezing is Fundamentally Stochastic due to Resistivity +Turbulence

* Nonideal effects of resistivity and viscosity in hydromagnetics can be represented by perturbation of Lagrangian particle paths with random Brownian motion. Magnetic flux is "frozen in" stochastically.

* As a consequence of a hydrodynamic effect—turbulent Richardson diffusion—any microscopic plasma mechanism of line diffusion will be enhanced in the presence of high-Reynolds-number turbulence.

* The turbulent dispersion at long times and large scales (inertial-range) is independent of the precise microscopic dissipation. Magnetic line-motion remains stochastic in the limit of large conductivity and small viscosity.

*Properties of Lagrangian particle-pair separation are critical to understand fast dynamo and fast reconnection in turbulent MHD plasmas.

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