Critical points of integrals of soliton equations

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Integrals of motion of periodic KdV

The periodic Korteweg-de Vries equation

$$4u_t = 6uu_x - u_{xxx}, \quad u(x+T) = u(x)$$

has an infinite number of integrals of motion of the form

$$H_n = \frac{1}{T}\int_0^T P_n(u, u', \ldots, u^{(n+1)})\, dx, \quad n = -1, 1, 3, 5, \ldots,$$

where the P_n are differential polynomials in u:

$$P_{-1} = -\frac{u}{2}, \quad P_1 = -\frac{u^2}{8}, \quad P_3 = -\frac{u_x^2 + 2u^3}{32}, \ldots$$

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Consider a linear combination $H = H_{2n+1} + c_{2n-1}H_{2n-1} + \ldots + c_{-1}H_{-1}$, $c_i \in \mathbb{R}$. The critical points of such a functional are solutions of the *n-th stationary KdV equation*.

Genus one solutions of KdV

For example, the critical points of the functional

$$H = H_3 - \frac{g_2 H_{-1}}{8} = -\frac{1}{32T}\int_0^T (u_x^2 + 2u^3 - g_2 u)dx$$

are stationary solutions of the usual KdV equation:

$$16\,\frac{\delta H}{\delta u} = u_{xx} - 3u^2 + g_2 = 0.$$

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We need to define the integrals H_n for arbitrary finite-gap solutions, which are singular, complex and quasi-periodic.

Extension of the Hamiltonians

The genus one solutions of KdV have the form

$$u(x) = 2\wp(x + x_0),$$

where $\wp(x) = \wp(x|\omega, \omega')$ is the Weierstrass $\wp$-function corresponding to the elliptic curve Γ:

$$Y^2 = E^3 - g_2 E - g_3 = (E - E_0)(E - E_1)(E - E_2).$$

The quasi-momentum differential dQ is defined by the properties:

$$dQ \sim \frac{dz}{z^2}, \quad z = E^{-1/2}, \quad \operatorname{Im} \oint_\gamma dQ = 0 \quad \forall \text{ cycle } \gamma.$$

The expansion of dQ at infinity gives the H_n:

$$Q = z^{-1} + \sum_{n=0}^{\infty} H_{2n-1}\, z^{2n+1}.$$

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A curve Γ is a critical point of $\mathcal{H}_3$ if and only if the differential YdE is *real-normalized*, i.e. if

$$\text{Im} \oint_\gamma YdE = 0 \quad \forall \text{ cycle } \gamma.$$

This is known as the Boutroux–Krichever condition. It has appeared in a number of physical problems, such as

- Hele-Shaw flows
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(Fucito, Gamba, Martellini, Ragnisco, 1992) There exists a unique curve Γ satisfying this condition:

$$g_3 = \frac{4h^2+1}{(4h^2-3)^{3/2}} g_2^{3/2}, \quad h \approx 3.246382$$

The moduli space $\mathcal{M}_{g,1}(n)$

Let $\mathcal{M}_{g,1}(n)$ denote the moduli space of smooth Riemann surfaces Γ with a marked point P and an $(n+1)$-jet of local coordinates $[z]$ at P. We define a collection of functions on $\mathcal{M}_{g,1}(n)$ and determine their critical points.

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Let E be the unique real-normalized Abelian integral with a pole at P of the form

$$E = z^{-n} + O(z) \text{ near } P, \quad \operatorname{Im} \oint_\gamma dE = 0 \quad \forall \text{ cycle } \gamma.$$

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Let z be the unique coordinate in the jet $[z]$ such that $E = z^{-n}$ exactly. Let $q(t) = q_m t^m + \cdots + q_0$ be a polynomial. There exists a unique real-normalized Abelian integral Q

$$Q = \sum_{j=0}^{m} q_j z^{-j} + O(z) \text{ near } P, \quad \operatorname{Im} \oint_\gamma dQ = 0 \quad \forall \text{ cycle } \gamma.$$

Coordinates and functions on $\mathcal{M}_{g,1}(n)$

Expand the differential QdE at P:

$$QdE = \left(\sum_{k=0}^{n+m} T_k z^{-k-1} + \sum_{j=1}^{\infty} H_j z^{j-1} \right) dz.$$

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Fix a basis of cycles A_i, B_i and consider the periods of dE and dQ

$$\tau^E_{A_i} = \oint_{A_i} dE, \quad \tau^E_{B_i} = \oint_{B_i} dE, \quad \tau^Q_{A_i} = \oint_{A_i} dQ, \quad \tau^Q_{B_i} = \oint_{B_i} dQ,$$

and the A-periods of QdE:

$$a_i = \oint_{A_i} QdE.$$

The functions T_k, $\tau^E_{A_i}$, $\tau^E_{B_i}$, $\tau^Q_{A_i}$, $\tau^Q_{B_i}$, a_i are coordinates on $\mathcal{M}_{g,1}(n)$, the functions H_j are the Hamiltonians that are of interest to us.

Hierarchy of foliations on $\mathcal{M}_{g,1}(n)$

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Let $\mathcal{L} \subset \Lambda$ denote a leaf along which the periods of both dE and dQ are constant:

$$\delta \oint_\gamma dE = \delta \oint_\gamma dQ = 0 \quad \forall \delta \in T_p(\mathcal{L}) \text{ and } \forall \text{ cycle } \gamma.$$

The extended Hamiltonians

We now define a real-valued function on $\mathcal{M}_{g,1}(n)$ and study its critical points on the leaves of the foliations defined above. Consider the expansion of QdE:

$$QdE = \left(\sum_{k=0}^{n+m} T_k z^{-k-1} + \sum_{j=1}^{\infty} H_j z^{j-1} \right) dz.$$

Let $c_j, d_j \in \mathbb{R}$ be real coefficients. Consider the following function on $\mathcal{M}_{g,1}(n)$:

$$\mathcal{H} = \sum_{j=1}^{M} \left(c_j \mathrm{Re} H_j + d_j \mathrm{Im} H_j \right).$$

Critical points on $\mathcal{H}$

Theorem (Krichever, Z.) Let D denote the intersection of the zero-divisors of dE and dQ.

A. The point $(\Gamma, P, [z])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}$ iff there exists a meromorphic function Y on Γ with a pole at P

$$Y = \sum_{j=1}^{\infty}(c_j - id_j)z^{-j} + O(z) \text{ near } P,$$

with poles at the divisor D, and no other singularities.

B. $(\Gamma, P, [z])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}_E$ iff

$$\operatorname{Im} \oint_\gamma Y dE = 0 \quad \forall \text{ cycle } \gamma.$$

C. $(\Gamma, P, [z])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}_Q$ iff

$$\operatorname{Im} \oint_\gamma Y dQ = 0 \quad \forall \text{ cycle } \gamma.$$

D. $(\Gamma, P, [z])$ is a critical point of $\mathcal{H}$ restricted to Λ iff Y satisfies both.

Proof

Consider a deformation $\delta \in T_p(\Lambda)$, and consider the

$$T_p(\Lambda) \to \Omega(\Gamma, P, E, Q), \quad \delta \mapsto (\delta Q|_E)dE$$

Here $\Omega(\Gamma, P, E, Q)$ is the space of multi-valued differentials on Γ having at most a simple pole at P and jumps along the cuts proportional to dE and dQ with real coefficients.

Theorem (Krichever, Phong, 1997) Let $D = \sum \mu_s Q_s$ be the intersection of the zero divisors of dE and dQ. Then the image of $T_p(\Lambda)$ under this homomorphism f is the subspace $\Omega_D(\Gamma, P, E, Q) \subset \Omega(\Gamma, P, E, Q)$ of differentials having zeroes at Q_s of orders μ_s. In particular, if the zero divisors of dE and dQ do not intersect, then this map is an isomorphism.

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$$\delta\mathcal{H} = \mathrm{Re}\ \mathrm{Res}_P\left(Y\delta Q|_E dE\right) = -\frac{1}{2\pi}\mathrm{Im}\left(\oint_{\partial\Gamma} Y\delta Q|_E dE\right).$$

Plugging in the possible $\delta Q|_E dE$, we get conditions **A**-**D**.

The moduli space $\mathcal{M}_{g,N}(\mathbf{n})$

Let $\mathbf{n} = (n_1, \ldots, n_N)$ be a multi-index. Let $\mathcal{M}_{g,N}(\mathbf{n})$ denote the moduli space of smooth Riemann surfaces Γ with marked points $P_1, \ldots, P_N$, an $(n_1 + 1)$-jet of local coordinates $[z_1]$ at P_1, and n_α-jets of local ccordinates $[z_\alpha]$ at P_α.

Let E be the real-normalized Abelian integral with poles at P_α of the form

$$E = z_1^{-n_1} + O(z_1) \text{ near } P_1, \quad E = z_\alpha^{-n_\alpha} + O(1) \text{ near } P_\alpha, \quad \operatorname{Im} \oint_\gamma dE = 0.$$

Let $q_\alpha(t)$ be polynomials of degrees $m_\alpha \leq n_\alpha$, let R_α^Q be imaginary numbers adding up to zero. Let Q be the real-normalized Abelian integral

$$Q = \sum_{j=1}^{m_\alpha} q_{\alpha,j} z_\alpha^{-j} + R_\alpha^Q \log z_\alpha + O(1) \text{ near } P_\alpha, \quad \operatorname{Im} \oint_\gamma dQ = 0$$

Coordinates on $\mathcal{M}_{g,N}(\mathbf{n})$

Consider the expansion of QdE at P_α:

$$QdE = \left(\sum_{k=0}^{n_\alpha + m_\alpha} T_{\alpha,k} z_\alpha^{-k-1} - n_\alpha R_\alpha^Q z^{-n_\alpha - 1} \log z_\alpha + \sum_{j=1}^{\infty} H_{\alpha,j} z_\alpha^{j-1} \right) dz_\alpha .$$

Consider the periods of the differentials dE and dQ:

$$\tau_{A_i}^E = \oint_{A_i} dE, \quad \tau_{B_i}^E = \oint_{B_i} dE, \quad \tau_{A_i}^Q = \oint_{A_i} dQ, \quad \tau_{B_i}^Q = \oint_{B_i} dQ,$$

and the A-periods of the differential QdE:

$$a_i = \oint_{A_i} QdE .$$

The main difference with the one-point case is that the $H_{\alpha,j}$ are no longer globally well-defined on $\mathcal{M}_{g,N}(\mathbf{n})$. However, their imaginary parts $\mathrm{Im}\ H_{\alpha,j}$ are well-defined.

Hierarchy of foliations on $\mathcal{M}_{g,1}(\mathbf{n})$

Let $\Lambda' \subset \mathcal{M}_{g,N}(\mathbf{n})$ be a leaf along which the functions $T_{\alpha,k}$ for $k = 1, \dots, m_\alpha + n_\alpha$ are constant:

$$\delta T_{\alpha,k} = 0 \quad \forall k > 0 \text{ and } \forall \delta \in T(\Lambda').$$

Let $\Lambda \subset \Lambda'$ be a leaf along which all the functions $T_{\alpha,k}$ are constant:

$$\delta T_{\alpha,k} = 0 \quad \forall k \geq 0 \text{ and } \forall \delta \in T(\Lambda).$$

Let $\mathcal{L}_E \subset \Lambda$ denote a leaf along which the periods of dE are constant:

$$\delta \oint_\gamma dE = 0 \quad \forall \delta \in T_p(\mathcal{L}_E) \text{ and } \forall \text{ cycle } \gamma.$$

Let $\mathcal{L}_Q \subset \Lambda$ denote a leaf along which the periods of dQ are constant:

$$\delta \oint_\gamma dQ = 0 \quad \forall \delta \in T_p(\mathcal{L}_Q) \text{ and } \forall \text{ cycle } \gamma.$$

Let $\mathcal{L} \subset \Lambda$ denote a leaf along which the periods of both dE and dQ are constant:

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The Hamiltonian $\mathcal{H}$ and its critical points

Let $d_{\alpha,j}$ for $\alpha = 1, \ldots, N$, $j = 1, \ldots, M_\alpha$ be real constants. The function

$$\mathcal{H} = \sum_{\alpha,j} d_{\alpha,j} \mathrm{Im} H_{\alpha,j}$$

is well-defined on $\mathcal{M}_{g,N}(\mathbf{n})$.

Theorem (Krichever, Z.) Let D denote the intersection of the zero-divisors of dE and dQ.

A. The point $(\Gamma, P_\alpha, [z_\alpha])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}$ iff there exists a meromorphic function Y on Γ with a pole at P

$$Y = i \sum_{j=1}^{M_\alpha} d_{\alpha,j} z_\alpha^{-j} + O(1) \text{ near } P_\alpha,$$

with poles at the divisor D, and no other singularities.

B. $(\Gamma, P_\alpha, [z_\alpha])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}_E$ iff

$$\mathrm{Im} \oint_\gamma Y dE = 0 \quad \forall \text{ cycle } \gamma.$$

C. $(\Gamma, P_\alpha, [z_\alpha])$ is a critical point of $\mathcal{H}$ restricted to $\mathcal{L}_Q$ iff

$$\operatorname{Im} \oint_\gamma Y dQ = 0 \quad \forall \text{ cycle } \gamma.$$

D. $(\Gamma, P_\alpha, [z_\alpha])$ is a critical point of $\mathcal{H}$ restricted to Λ iff Y satisfies both.
E. $(\Gamma, P_\alpha, [z_\alpha])$ is a critical point of $\mathcal{H}$ restricted to Λ' iff Y also satisfies the condition

$$Y = i \sum_{j=1}^{M_\alpha} d_{\alpha,j} z_\alpha^{-j} + O(z_\alpha) \text{ near } P_\alpha.$$