On energy concentration and the size of the Navier– Stokes singular set

Walter Craig

Department of Mathematics & Statistics

McMaster University

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Joint work with:

Andrei Biryuk	University of Krasnodar
Maxim Arnold	International Institute of Earthquake Prediction Theory and Mathematical Geophysics RAS Moscow

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Outline

- A study of the situation in which $t \mapsto u(\cdot, t)$ is not continuous in L^2 (in the strong topology)
- The singular sets $S(u)$ and the L^2 concentration set $WF^{L^2}(u)$
- Three estimates on Leray weak solutions
- Lower bounds on the concentration rate to $WF^{L^2}(u)$
- Microlocal defect measures, and ideas of the proof

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Navier – Stokes equations

The equations of motion of an incompressible viscous fluid

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u \\ \nabla \cdot u &= 0 \end{aligned} \tag{1}$$

Initial data

$$u(x,0) = u_0(x) \ , \ \nabla \cdot u_0 = 0$$

Space-time domain

$$D = \mathbb{R}^3 \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ := Q$$

Alternatively

$$D = \mathbb{T}^3 \qquad (x,t) \in \mathbb{T}^3 \times \mathbb{R}^+ = Q$$

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Leray weak solutions

The usual definition of a weak solution over $t \in [0, T]$:

1. The pair $(u(x,t), p(x,t))$ is a distributional solution of (1)

2. Integrability conditions

$$u \in L^{\infty}([0,T]; L^2(D)) \cap L^2([0,T]; \dot{H}^1(D)) , \tag{2}$$

$$p \in L^{5/3}_{loc}(Q) \qquad (Sohr \; \& \; vonWahl \; (1986))$$

3. The energy inequality is satisfied

$$\tfrac{1}{2} \int_D |u(x,t)|^2 \, dx + \nu \int_0^t \int_D |\nabla u(x,s)|^2 \, dxds \le \tfrac{1}{2} \int_D |u_0(x)|^2 \, dx \tag{3}$$

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Definition (Singular set)

Given a weak solution (u, p) of (1), the singular set $S(u) \subseteq Q$ is the set of space-time points at which $u(x, t)$ is not locally bounded.

That is, $(x_0, t_0) \notin S(u)$ if there is a neighborhood $B_r(x_0, t_0)$ such that

$$u \in L^\infty(B_r(x_0, t_0)) \tag{4}$$

This makes sense due to a theorem of Serrin (1962) which implies that if $(x_0, t_0) \notin S(u)$, then for all k (and with some $0 < \alpha < 1$)

$$\partial_x^k u(x, t) \in C^\alpha(B_{r/2}(x_0, t_0)) \tag{5}$$

Serrin's condition (4) is actually $u \in L^s L^p(B_r(x_0, t_0))$ for $\frac{3}{p} + \frac{2}{s} < 1$

Improved by Struwe (1995) to equality, with $s < \infty$

And by Escauriaza, Seregin and Sveràk (2003) to $u \in L^\infty L^3(Q_r(x_0, t_0))$, for Q_r parabolic cylinders

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Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ *divergence free, then there exists at least one weak solution to* (1) *globally in time. Weak solutions satisfy*

$$u \in L^\infty([0,T]; L^2(D)) \ \cap \ L^2([0,T]; \dot{H}^1(D)) \quad p \in L^{5/3}_{loc}(Q) \tag{6}$$

A lot is known about such solutions, including weak continuity

$$u \in C_t(L^2_x : \text{weak} \quad \text{topology})$$

as well as

$$u \in L^s_t(L^p_x) \,, \qquad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \qquad 2 \le p \le 6$$

Uniqueness and global regularity are unknown

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Upper bounds on the singular set

- The singular set $S(u)$ is closed, by definition
- The set of singular times $\tau(u) = \pi_t S(u) \in \mathbb{R}^+$ has zero $1/2$-Hausdorff dimensional measure $(Leray, \ Foias \,\&\, Temam)$

$$\mathcal{H}^{1/2}(\tau(u)) = 0 \tag{7}$$

Theorem (partial regularity Caffarelli, Kohn & Nirenberg (1982))

If (u, p) is a suitable weak solution of (1) *then the parabolic one-dimensional Hausdorff measure of $S(u)$ is zero;*

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Restrict to a time slice

- For T fixed, the singular set $S_T := S(u) \cap \{t = T\}$ in each time slice is at most one-dimensional
$$\mathcal{H}^1(S_T) = 0 \tag{9}$$

- Suitable weak solutions are those satisfying a local energy inequality

$$\int_D \tfrac{1}{2}|u(\cdot,t)|^2 \varphi \, dx \Big|_{t=0}^{T} + \nu \int_0^T \int_D |\nabla u(\cdot,t)|^2 \varphi \, dxdt \tag{10}$$
$$\le \tfrac{1}{2} \int_0^T \int_D |u(\cdot,t)|^2 \Big(\partial_t \varphi + \nu \Delta \varphi \Big) \, dxdt$$
$$+ \int_0^T \int_D \Big(p + \tfrac{1}{2}|u(\cdot,t)|^2 \Big) u \cdot \nabla \varphi \, dxdt$$

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Three inequalities

1. The energy inequality

$$\frac{1}{2}\int_D |u(x,t)|^2\,dx + \nu\int_0^t\int_D |\nabla u(x,s)|^2\,dxds \le \frac{1}{2}\int_D |u_0(x)|^2\,dx$$

and its consequences under interpolation

2. Foiaş, Guillopé & Temam (1981), Chemin (2004)

$$\int_0^T \|u(\cdot,t)\|_{L^\infty}\,dt < +\infty\,, \qquad \forall T\in\mathbb{R}^+ \tag{11}$$

Theorem (Biryuk & C. (2008))

Let $B_R(0)\subseteq L^2(D)$, and define

$$A_{R_1} := \{(\hat{u}(\xi))_{\xi\in\mathbb{R}^3} \,:\, |\xi||\hat{u}(\xi)| < R_1\} \tag{12}$$

If $R^2/\sqrt{2\pi}^3 < \nu R_1$ then $A_{R_1}\cap B_R(0)$ is a (future) invariant set for Navier – Stokes flow.

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- That is to say, if initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$ then for all time $t > 0$

$$|\hat{u}(\xi, t)| < \frac{R_1}{|\xi|}\,, \qquad \forall \xi \tag{13}$$

Corollary (B & C (2008))

If the initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$, then additionally for all $T \geq 0$

$$\nu \int_0^T |\hat{u}(\xi, s)|^2 \, ds \leq \frac{R_2^2}{|\xi|^4} \tag{14}$$

- Proof of the theorem and corollary given at end of talk if there is time.

  The quantity $\sup_t \| |\xi| \hat{u}(\xi, t) \|_{L^\infty}$ scales like the BV norm $\sup_t \|\partial_x u(\cdot, t)\|_{L^1}$ (for which there are no known bounds).

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Energy spectral function

- The estimate (13) has consequences on the behavior of the energy spectral function

$$E(k,t) := \int_{\mathbb{S}^{d-1}:|\xi|=k} |\hat{u}(\xi,t)|^2 \, dS(\xi) \tag{15}$$

in dimensions $d = 2$ and $d = 3$

- Indeed when $d = 3$ a uniform upper bound

$$\begin{aligned} 0 \quad &\leq \quad E(k,t) = \int_{\mathbb{S}^2:|\xi|=k} |\hat{u}(\xi,t)|^2 \, dS(\xi) \\ &\leq \quad 4\pi k^2 \times \frac{R_1^2}{k^2} = 4\pi R_1^2 \end{aligned} \tag{16}$$

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- When $d = 3$ the time average of $E(k,t)$

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Contrast this with the Kolmogorov exponent

$$E(k) \sim C\frac{1}{k^{5/3}}$$

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- The energy $\frac{1}{2}\|u(\cdot,t)\|^2_{L^2}$ could be discontinuous at $T \in \tau(u)$.
- Or else it may be continuous, but nonetheless $\|\nabla u(\cdot,t)\|^2_{L^2}$ is unbounded on $[T-\delta, T]$ for any δ
- Decompose the set of singular times $\tau(u)$ into

$$\tau(u) := \tau_1 \cup \tau_2$$

where τ_1 are the energy discontinuities

$$\tau_1 := \{T : \limsup_{t \to T^-} \|u(\cdot,t)\|^2_{L^2} > \|u(\cdot,T)\|^2_{L^2}\}$$

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$$\tau(u) := \tau_1 \cup \tau_2$$

where τ_1 are the energy discontinuities

$$\tau_1 := \{T : \limsup_{t \to T^-} \|u(\cdot,t)\|_{L^2}^2 > \|u(\cdot,T)\|_{L^2}^2\}$$

and $\tau_2 = \tau(u) \backslash \tau_1$ is the remainder

Lower bounds on the singular set

Theorem (Arnold & C. (2010))

If $T_0 \in \tau_1$ is an energy discontinuity for u then

$$\text{“dim”}\ (WF(u)) \geq 1 \tag{18}$$

- Dimension count:
  $S(u) \cap \{t = T_0\} := S_{T_0}(u)$ is a subset of $\mathbb{R}^3$
  while the wave front set $WF(u)$ is a subset of $S^*(\mathbb{R}^3)$, which is 5 dimensional

Outline of the result

- Identify the L^2 concentration set $S^{L^2}(u) \subseteq S(u)$ as a geometric subset of $\mathbb{R}^3$
- Probe the solution with Weyl calculus pseudodifferential operators
- Identify the L^2 discontinuities with a defect measure, and $WF^{L^2}(u)$ with its support
- Argue by contradiction, using the estimate (13) and the dominated convergence theorem, if the solution concentrates onto $WF^{L^2}(u)$ on a set which is too small

The L^2 concentration set

- Suppose that $T_0 \in \tau_1$, when $u(x,t)$ is not strong L^2 continuous.

Definition (L^2 concentration set)

The point $x_0 \notin S_{T_0}^{L^2}$ if there exists $r > 0$ such that

$$\lim_{t \to T_0^-} \|u(\cdot,t)\|^2_{L^2(B_r(x_0))} = \|u(\cdot,T_0)\|^2_{L^2(B_r(x_0))} \tag{19}$$

Thus L^2 concentration is associated with a point set $S_{T_0}^{L^2} \in \mathbb{R}^3$

- The set $S_{T_0}^{L^2}$ is closed
  Indeed norm convergence plus weak convergence implies strong convergence, and any $B_{r_1}(x_1) \subseteq B_r(x_0)$ shares this strong convergence
- Since $u(\cdot,t)$ is smooth outside the singular set $S_{T_0}(u)$,

$$S_{T_0}^{L^2} \subseteq S_{T_0}(u) \tag{20}$$

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Local behavior in x and ξ

- The Fourier transform $\hat{u}(\xi, t) \in Lip$ as a function of time
- Take $x_0 \in S^{L^2}_{T_0}$, and localize the convergence question

$$v(x,t) := (u(x,t) - u(x,T_0))\varphi(x) \; , \qquad 0 \le \varphi \in C_0^\infty(B_r(x_0)) \tag{21}$$

Proposition

As $t \to T_0^-$ then $\hat{v}(\xi, t) \to 0$ pointwise in $\xi \in \mathbb{R}^3$

- What causes a lack of strong convergence is loss of L^2 mass at $|\xi| \to \infty$

Weyl calculus

- Given a point $x_0 \in S^{L^2}_{T_0}$ we test for energy concentration using the Weyl pseudodifferential calculus

$$a^w(x,D)v(x,t) = \iint e^{i\xi\cdot(x-y)} a(\tfrac{x+y}{2},\xi)v(y,t)\,dyd\xi \tag{22}$$

for $a(x,\xi) \in S^0_{\rho\delta}$.

- A microlocal test of the energy is

$$\langle v \,|\, a^w(x,D)v\rangle = \iint a(x,\xi) W[v](x,\xi)\,dxd\xi \tag{23}$$

where $W[v]$ is the Wigner transform of v

$$W[v](x,\xi) := \frac{1}{(2\pi)^d} \int e^{i\xi\cdot y} v(x+y/2)\overline{v}(x-y/2)\,dy \tag{24}$$

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microlocal L^2 defects

- The operators $a^w(x,D)$ are L^2 bounded for $a(x,\xi) \in S^0_{\rho\delta}$
  Whenever $u(\cdot,t)$ converges strongly to $u(\cdot,T_0)$ on $B_r(x_0)$, for $v(x,t) := (u(x,t) - u(x,T))\varphi(x)$ then
$$\lim_{t\to T_0^-} \langle a \,|\, W[v(t)]\rangle = 0 \tag{25}$$
- However if $u(\cdot,t)$ converges weakly but not strongly to $u(\cdot,T_0)$, it is detected by a microlocal defect measure $\mu \in \mathcal{M}(S^*(\mathbb{R}^3))$.

Theorem (L. Tartar (1990), P. Gérard (1991))

Let μ_{x_0} be a microlocal defect measure of $\lim_{t\to T_0^-} v(x,t)$. Suppose that for all symbols $a \in S^0_{10}$ homogeneous degree zero
$$\lim_{t\to T_0^-} \langle a \,|\, W[v(t)]\rangle := \iint_{S^*(\mathbb{R}^3)} a(x,\xi)\mu_{x_0}(dxdS_\xi) = 0 \tag{26}$$
then $\mu_{x_0} = 0$ and $u(\cdot,t)$ converges strongly in $L^2(B_r(x_0))$ to $u(\cdot,T_0)$.

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the L^2 wave front set WF^{L^2}

Definition

Let $\mathcal{A}$ be the set of all microlocal defect measures μ_{x_0} of v as $t \to T_0^-$. Define the L^2 wave front set

$$WF^{L^2}_{x_0 T_0} := \overline{\bigcup_{\mathcal{A}} \operatorname{supp}(\mu_{x_0})} \tag{27}$$

the $S^0_{\rho\delta}$ Hörmander symbol classes and WF^{L^2}

- One probes the set $WF^{L^2}_{x_0 T_0}$ with symbols $a \in S^0_{\rho\delta}$, with $0 \le \delta < \rho \le 1$

  For example $a(x,\xi) = \varphi(x)\chi(\frac{\xi'}{\langle \xi_1 \rangle^\rho}) \in S^0_{\rho 0}$ for $0 < \rho \le 1$

  By standard theory, the operators $a^w(x,D)$ are bounded on L^2.

- Suppose that $0 \le a(x,\xi) \le 1$ is such that

$$\lim_{t \to T_0^-} \langle v | (1-a)^w(x,D) v \rangle = 0 \tag{28}$$

  then

$$WF^{L^2}_{x_0 T_0} \subseteq \text{supp}\,(a)$$

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L^2 concentration at infinity

- Given test symbols $a(x,\xi)$, the volume growth of $\mathrm{supp}\,(a)$ gives an upper bound on the size of the set of L^2 concentration in neighborhoods of $WF^{L^2}_{x_0 T_0}$.
- Consider $a(x,\xi) \in S^0_{\rho\delta}$ such that $0 \leq a \leq 1$,

$$\lim_{t \to T_0^-} \langle (1-a) \,|\, W[v(t)] \rangle = 0 \tag{29}$$

and

$$\mathrm{vol}\left(\pi_\xi \mathrm{supp}\,(a) \cap B_R(0)\right) \sim R^{1+\beta} \tag{30}$$

Definition (size of the concentration set for $WF^{L^2}_{x_0 T_0}$)

$$\bar{\beta}_{x_0}(v) := \inf(\beta) \tag{31}$$

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proof of the theorem that $A \cap B_R(0)$ is invariant

- For fixed ξ the field $\hat{u}(\xi) \in \mathbb{C}^2_\xi \subseteq \mathbb{C}^3$
  Because of incompressibility $\xi \cdot \hat{u}(\xi) = 0$

Proposition

The function $\hat{u}(\xi, t)$ is Lipschitz continuous as a function of t for every ξ

- The Fourier transform satisfies

$$\begin{aligned} \partial_t \hat{u}(\xi) &= -\nu|\xi|^2\hat{u}(\xi) - \frac{i\xi}{\sqrt{(2\pi)^3}}\Pi_\xi \int \hat{u}(\xi - \xi_1) \cdot \hat{u}(\xi_1)\, d\xi_1 \\ &:= X(u)_\xi \end{aligned} \tag{33}$$

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- Suppose that $\|u(\cdot)\|_{L^2} \leq R$ and consider the vector field $X(u)_\xi$ when $|\hat{u}(\xi)| = R_1/|\xi|$. The radial component is

$$\mathrm{re}(\hat{u}(\xi) \cdot X(u)_\xi) < -\nu|\xi|^2(R_1/|\xi|)^2 + (R_1/|\xi|)|\xi|\frac{R^2}{\sqrt{(2\pi)^3}} \qquad (34)$$

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- A fact about the vector field $X(\hat{u})$ is that solutions obey

$$|\hat{u}(\xi,T)|^2 - |\hat{u}_0(\xi)|^2 + 2\nu \int_0^T |\xi|^2 |\hat{u}(\xi,t)|^2\,dt$$
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(again setting $f = 0$ for simplicity)

- Writing $I^2(\xi) = \nu \int_0^T |\xi|^4 |\hat{u}(\xi,t)|^2\,dt$ this gives an inequality

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Proof of main theorem

Taking into account the local character of the problem, the real result states.

Conjecture (A & C (work in progress))

Suppose that $\bar{\alpha}$ *is the Hausdorff dimension of the set* $S_{T_0}^{L^2}$*, then*

$$\bar{\beta}_{x_0}(v) \geq \bar{\alpha} + 1 \tag{37}$$

- Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$\lim_{t \to T_0^-} \langle a | W[v] \rangle = \lim_{t \to T_0^-} \int v a^w(x, D) v \, dx \tag{38}$$
$$= \lim_{t \to T_0^-} \int d\eta \Big[\int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t) \overline{\hat{v}(\xi - \eta/2, t)} \, d\xi \Big]$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of a with respect to x.

- For each η, the volume growth of $\operatorname{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$
- There is now a majorant: for each η

$$|\tilde{a}(\eta, \xi)| |\hat{v}(\xi + \eta/2, t)| |\hat{v}(\xi - \eta/2, t)| \le |\tilde{a}(\eta, \xi)| \langle \xi + \eta/2 \rangle^{-1} \langle \xi - \eta/2 \rangle^{-1}$$

- This is integrable over ξ. Indeed

$$\int_\xi (*) \, d\xi \le \int \langle r \rangle^{-2} r^\beta \, dr < +\infty \tag{39}$$

The dominated convergence theorem implies that as $t \to T_0^-$ the limit vanishes.

- Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

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Happy 70th birthday Volodya. **Thank you**