New closed form analytic patterns of the quintic Ginzburg-Landau equation in one spatial dimension

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Outline

Motivation, physical problem, mathematical problem

Nonintegrable counting. The general analytic solution

Elliptic functions = the simplest nonlinear special functions

Existing methods (sufficient)

Subequation method (necessary)

Examples and results

Conclusion & perspectives

Problem (math): to find the "general analytic solution"

N:= differential order of the ODE.

- **Integrable** ODE (Painlevé property). Pb is: to find the general solution (N arb constants). Discard this case.
- **Partially integrable** ODE (fails the P test). The general solution, if it exists, has movable multivaluedness.

  Definition: **general analytic solution** : = the largest, M-parameter particular sol without movable critical singularities, $M \le N - 1$.

PROBLEM: to find this general analytic solution in closed form. Equivalent to find the largest subequation with the PP.

SOLUTION: IF this solution is elliptic or degenerate of elliptic, $\exists$ method (RC, M. Musette, Stud. Appl. Math. 123 (2009) 63–81) which finds **all** those solutions in closed form.

Integrable case. Example Korteweg-de Vries

$$U''' - \frac{6}{a} U U' = 0. \tag{1}$$

Skill: Total derivative $\Rightarrow$ first integral g_2:

$$U'' - \frac{3}{a} U^2 + a g_2 = 0. \tag{2}$$

Skill: Integrating factor U' $\Rightarrow$ first integral g_3:

$$\frac{1}{2} {U'}^2 - \frac{1}{a} U^3 + a g_2 U + 2a^2 g_3 = 0. \tag{3}$$

General solution is elliptic (doubly periodic)

$$U(\xi) = 2a\wp(\xi - \xi_0, g_2, g_3), \text{ arbitrary} = \xi_0, g_2, g_3. \tag{4}$$

The above integration relies on skill, but **skill is not a method**.

Partially integrable case. Examples

By definition, the equations fail the Painlevé test.

- Complex Ginzburg-Landau equation (CGL3)

$$iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0,\ pq\gamma \ne 0,\ (A, p, q) \in \mathcal{C},\ \gamma \in \mathcal{R}$$

(a generic amplitude equation, optical fibers, etc).

- Kuramoto-Sivashinsky equation (KS)

$$u_t + \nu u_{xxxx} + b u_{xxx} + \mu u_{xx} + u u_x = 0,\ \nu \ne 0,$$

(flame on a vertical wall, phase equation of CGL3).

Particular travelling waves are known.
They are all either elliptic (KS, $b^2 = 16\mu\nu$) or polynomial in $\tanh(k/2)(\xi - \xi_0)$.
Q.: Are there other travelling waves?
A.: *A priori* yes.
Problem: to find them all.

Outline

General analytic solution, local representation. KS

$$\nu u''' + bu'' + \mu u' + u^2/2 + A = 0,\ \text{Fuchs indices} = -1, \frac{13 \pm i\sqrt{71}}{2}.$$

Particular one-parameter sol. = Laurent series (local) $\chi = \xi - \xi_0$

$$u^{(0)} = 120\nu\chi^{-3} - 15b\chi^{-2} + \frac{15(16\mu\nu - b^2)}{4 \times 19\nu}\chi^{-1} + \frac{13(4\mu\nu - b^2)b}{32 \times 19\nu^2} + \dots,$$

General three-parameter solution (*illusory*, irrational Fuchs ind.):

$$\begin{aligned} u(\xi - \xi_0, \varepsilon c_+, \varepsilon c_-) \quad = \quad & u^{(0)} \\ & + \varepsilon[c_{-1}\chi^{-1}\text{Regular}(\chi) \\ & + \ c_+\chi^{(13+i\sqrt{71})/2}\text{Regular}(\chi) \\ & + \ c_-\chi^{(13-i\sqrt{71})/2}\text{Regular}(\chi)] + \mathcal{O}(\varepsilon^2)\}. \end{aligned}$$

c_{-1} can be set to 0 (Poincaré).
Reachable constants$=3-2=1=$ the origin ξ_0 of ξ.
General analytic solution = closed form expression for $u^{(0)}$=?

Outline

Elliptic functions: the simplest nonlinear special fns

Which functions can be defined by a differential eq. (exclude Γ)?

Any **linear** ODE defines a function (because its general solution is uniformizable) : $u^{(N)} = 0$ (polynomials, rational functions), $u' = u$ (exponential, trigo, hyperbolic trigo), $u'' + p(x)u' + q(x)u = 0$ (Airy, Bessel, Hermite, Whittaker, Gauss, Heun ...), ...

Nonlinear (nonlinearizable) ODEs? Much more difficult, but systematic (Painlevé). The result is (Fuchs, Poincaré, Painlevé):

Order one $F(u', u, x) = 0$: one function (modulo some group), **the** elliptic function.

Order two $F(u'', u', u, x) = 0$: the six functions of Painlevé.

To summarize:

The elliptic function(s) is (are) the simplest nonlinear "special" function(s).

Elliptic functions: their successive degeneracies

Good book: Abramowitz and Stegun (1972). Bad book: Byrd and Friedman (1954).

Any elliptic f. is a rational f. of $\wp, \wp'$ (birational group).
Canonical representative = the choice of Weierstrass,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Elliptic f^{ns} have two successive degeneracies: rational functions of one exponential $e^{k(z-z_0)}$ (degeneracy of Weierstrass ODE to constant coeff Riccati), then rational functions of $z - z_0$:

$$\wp(z) = \begin{cases} \text{doubly periodic ("cnoidal wave")}, \ e_j \text{ distinct}, \\ \text{simply periodic } = 3q\coth^2(\sqrt{3q}z) - 4q, \ e_1 = e_2 \neq e_3 \\ \text{rational } = \dfrac{1}{z^2}, \ e_1 = e_2 = e_3. \end{cases}$$

Practically. The Weierstrass first order ODE (and its degeneracy the Riccati ODE) are **the simplest nonlinear elementary building blocks** to build (particular) solutions to higher order autonomous nonlinear ODEs.

Elliptic functions, a few patterns

Abramowitz and Stegun Fig. 16.1

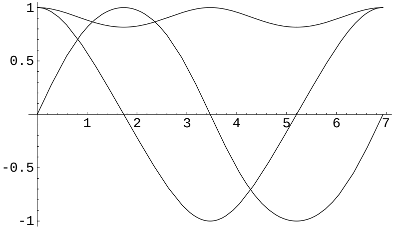
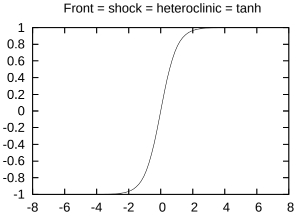
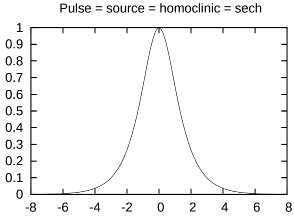


Figure: The copolar trio $(\mathrm{sn}, \mathrm{cn}, \mathrm{dn})$ of Jacobi, over one real period, for the value $k = 1/3$. The curves of sn, cn are close to those of $\sin, \cos$, and dn never vanishes.





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Existing methods (sufficient)

Mostly 1983, Weiss, Tabor and Carnevale.

1. Assume a **given class of expressions** for the general analytic solution,
2. Check whether there are indeed solutions in that class.

Various names: truncation method, tanh method, exp method, Jacobi expansion method, "new" or "extended" method, etc.

Examples of classes:

polynomial in tanh (Weiss et al 1983 and followers),
polynomial in $\wp, \wp'$ (Samsonov, Kudryashov 1989),
polynomial in tanh, sech (RC and M. Musette 1992, Pickering 1993).

All these methods are **sufficient** (street lamp), they cannot find any solution outside the given class.

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Two results of Briot and Bouquet

Théorie des fonctions elliptiques (Mallet-Bachelier, Paris, 1859).
http://gallica.bnf.fr/document?O=N099571

Elliptic order:= nb of poles per period parallelogram.

Theorem (BB). Given elliptic functions u, v with same periods, $\exists F$:

$$F(u,v) \equiv \sum_{k=0}^{m} \sum_{j=0}^{n} a_{j,k} u^j v^k = 0, \tag{22}$$

$\deg(F,u) = \mathrm{order}(v) = n$, $\deg(F,v) = \mathrm{order}(u) = m$. If $v = u'$,

$$F(u,u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j {u'}^k = 0,\ a_{0,m} \neq 0. \tag{23}$$

Theorem (BB, Poincaré, Painlevé). If a first order m-th degree autonomous algebraic ODE has the Painlevé property,

- it must have the form (23),
- its general solution $u = f(x - x_0)$ is either elliptic (two periods) or rational(e^{kx}) (one period) or rational (no period).

General method to find all elliptic solutions

M. Musette and RC, Physica D 181 (2003) 70–79. nlin.PS/0302051;
RC and M. Musette, *The Painlevé handbook* (Springer, 2008);
RC and M. Musette, Stud. Appl. Math. 123 (2009) 63–81. arXiv:0903.2009

4 ingredients: 2 theorems of Briot and Bouquet, Laurent series, algorithm of Poincaré.

Input: N-th order ($N \geq 2$) any degree autonomous algebraic ODE admitting a Laurent series.

Output: all its elliptic and degenerate elliptic solutions in closed form.

Steps:

1. Find the structure of singularities (e.g., 4 simple poles, 2 double poles). Deduce the elliptic orders m, n of u, u'.
2. Compute slightly more than $(m+1)^2$ terms in Laurent,

$$u = \chi^p \sum_{j=0}^{J} u_j \chi^j + \mathcal{O}(\chi^{J+p+1}),\ \chi = \xi - \xi_0. \quad (24)$$

3. Define the first order m-th (or divisor) degree subequation

$$F(u,u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j {u'}^k = 0,\ a_{0,m} \ne 0,\ 2m-2k \le n. (25)$$

4. Require each Laurent series (24) to obey $F(u,u')=0$,

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right),\ \forall j\ :\ F_j = 0.\ (26)$$

   and solve this **linear overdetermined** system for $a_{j,k}$.
5. Integrate each resulting ODE $F(u,u')=0$, algorithm of Poincaré. Mark van Hoeij, package "algcurves", Maple V

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Tutorial example, KdV

$$U''' - (6/a)UU' = 0. \tag{27}$$

One double pole $U \sim 2a\chi^{-2}, \chi' = 1$. Elliptic orders $(U) = 2, (U') = 3$.

$$U = 2a\chi^{-2} + U_4\chi^2 + U_6\chi^4 + \frac{U_4^2}{6a}\chi^6 + \dots, \text{ (9 terms)}, \ U_4, U_6 = \text{arbitrary},$$

$$F \equiv U'^2 + a_{0,1}U' + a_{1,1}UU' + a_{0,0} + a_{1,0}U + a_{2,0}U^2 + a_{3,0}U^3, a_{0,2} = 1,$$

Step 4. Linear overdetermined system (26),

$$\begin{cases} F_0 \equiv 16a^2 a_{0,2} + 8a^3 a_{3,0} = 0, \ a_{0,2} = 1, \\ F_1 \equiv -8a^2 a_{1,1} = 0, \\ F_2 \equiv 4a^2 a_{2,0} = 0, \\ F_3 \equiv -4a a_{0,1} = 0, \\ F_4 \equiv 2a a_{1,0} - 16a a_{0,2} U_4 + 12a^2 a_{3,0} U_4 = 0, \\ F_5 \equiv 0, \\ F_6 \equiv a_{0,0} + 4a a_{2,0} U_4 - 32a a_{0,2} U_6 + 12a^2 a_{3,0} U_6 = 0, \\ \dots \end{cases} \tag{28}$$

Only one solution

$$U'^2 - (2/a)U^3 + 20U_4U + 56aU_6 = 0. \qquad (29)$$

(Arbitrary constants U_4, U_6) $\Leftrightarrow$ (first integrals).

Systematic, no skill required to find first integrals.

Step 5 (algorithm of Poincaré). Establish the birational transformation $(U, U') \to (\wp, \wp')$. Maple statements:
with(algcurves);
genus((29),U, U');
Weierstrassform((29),U, U', wp, wpprime);

Example. Complex quintic Ginzburg-Landau

$$iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0,\ \Im(p/r) \neq 0,$$

$$(p,q,r) \in \mathcal{C},\ \gamma \in \mathcal{R},\ A = \sqrt{M(\xi)}e^{i(-\omega t + \varphi(\xi))},\ \xi = x - ct.$$

Only three solutions were known:

1. Heteroclinic front or shock (van Saarloos, Hohenberg 1992)

$$A = A_0\left((k/2)(\tanh k\xi/2 + \varepsilon)\right)^{1/2} e^{i[\alpha \log\cosh k\xi/2 + K\xi - \omega t]},\ \varepsilon^2 = 1$$

2. Homoclinic source or sink (Marcq, Chaté, RC 1994) (HS 1972 for $\cosh ka = 0, r_0 = 0$; vSH 1992 for $r_0 = 0$)

$$A = A_0\left(\frac{k\sinh ka}{\cosh k\xi + \cosh ka} + r_0\right)^{1/2} e^{i[\alpha\log(\cosh k\xi + \cosh ka) + K\xi -$$

3. Elliptic solution (Vernov 2007)

$$\begin{cases} M = c_0\dfrac{4\wp^2(\xi) - g_2}{4\wp^2(\xi) + g_2},\ \varphi' = (c/2)\Re(1/p) + \left(-\dfrac{3g_2}{4\wp(2\xi)}\right)^{1/2}, \\ g_3 = 0,\ g_2 = -\dfrac{g_r^2}{27},\ c_0^2 = \dfrac{4g_r}{3\Im(r/p)},\ q = \Re(r/p) = c\Im(p) = 0. \end{cases}$$

Complex quintic Ginzburg-Landau. Subequation method

The ODE system in (M, φ') has third order. Discard φ' (slave var).
M has four simple poles $M \sim r\chi^{-1}$, $e_i^2 r^4 - 4 e_r r^2 - 3 = 0$,
$e_r + i e_i = r/p$. Ellipt. orders$(M, M') = (4, 8)$.

Table: CGL5. Solutions ordered by nb of Laurent series

nb(Laurent)	genus	periods	pattern	codim	Ref
1	0	1	front	2	van Saarloos Hoh. 1992
2, $r_1 + r_2 = 0$	0	1	source	3	Marcq, Chaté, RC 1994
2, $r_1 + r_2 = 0$	1	2	?	3	new
2, $r_1 + r_2 \neq 0$					in progress
3					in progress
4	1	2	?	5	Vernov 2007
4	1	2	?	4	new

For $c\Im(p) = 0$ the last sol reduces to the solution of Vernov 2007:

$$(3M')^4 - M^2(3e_i M^2 - 4\sigma_r)^3/e_i = 0,\ 9\psi'^2 - 12\psi^4 - g^2 = 0$$

Complex quintic Ginzburg-Landau. One new solution

$q=0$, $\Re(r/p)=0$, $g_i=-(3/16)c_{\mathrm{i}}^2$, $c_{\mathrm{i}}\equiv\Im(c/p)$, arb.$=(g_r,c_{\mathrm{i}})$

$$F\equiv {M'}^4-2c_{\mathrm{i}}M{M'}^3+\frac{1}{2e_i}c_{\mathrm{i}}^2{M'}^2(3e_iM^2-g_r)+\frac{3^4c_{\mathrm{i}}^8}{2^{12}e_i^2}$$
$$+\frac{c_{\mathrm{i}}^4}{2^5e_i^2}\left(2g_r^2+6e_ig_rM^2-9e_i^2M^4\right)-\frac{1}{3^4e_i}M^2\left(3e_iM^2-4g_r\right)^3=0.$$

Step 5. (Briot-Bouquet) $M=\dfrac{P_4(\wp)+P_2(\wp)\wp'}{P_4(\wp)}=$ ugly.
Maple **Weierstrassform** command fails to integrate for arbitrary (g_r,c_{i}), it only succeeds for numerical $(g_r,c_{\mathrm{i}})=$ ugly BB form.

Results for various partially integrable PDEs

$$
\begin{aligned}
&\text{KdV} \ : \ U''' - (6/a)UU' = 0, \\
&\text{KS} \ : \ \nu u''' + bu'' + \mu u' + u^2/2 + A = 0, \\
&\text{CGL3} \ : \ iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0, \ qp \neq 0, \ \Im(q/p) \neq 0, \\
&\text{CGL5} \ : \ iA_t + pA_{xx} + q|A|^2 A + r|A|^4 A - i\gamma A = 0, \ pr \neq 0, \ \Im(p/r) \\
&\text{Chazy III} \ : \ u''' - 2uu'' + 3u'^2 = 0.
\end{aligned}
$$

PDE	poles	m	$(m+1)^2-1$	$a_{j,k}$	u_k	solutions	Ref
KdV	1P2	2	8	6	8	1 ell	
Kuramoto	1P3	3	15	10	16	1 ell + 4 trig	MC 2003
CGL3	2P2	4	24	18	24	6 trig + 1 rat	MC 2003; Hone 20
CGL5	4P1	4	24	24		1 ell + n trig	Vernov 2007
2 CGL3	2P2	4	24	18		To be done	RC, Musette 2000
Chazy III	1P2	2	8	6		1 rational	
Bureau	1P2	2	8	6		1 rational	

Conclusion, perspectives

Conclusion

- **All** elliptic and degenerate sol (trigo, rational) are found.
- Includes all "truncation", "extended", "new", etc methods.
- Makes these methods obsolete.
- For constant coeff. ODEs, *linear* complexity.

Current and future work

- Revisit physically interesting PDEs with insufficient solutions.
- Find general analytic solutions which are not elliptic (e.g. doubly periodic of second or third species, Hermite, Halphen).

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