



Non-Equilibrium Statistical Physics of Currents in Queuing Networks

Michael Chertkov

Center for Nonlinear Studies & Theory Division, LANL

Joint work with

V. Chernyak, D.A. Goldberg and K. Turitsyn

UA, Tucson

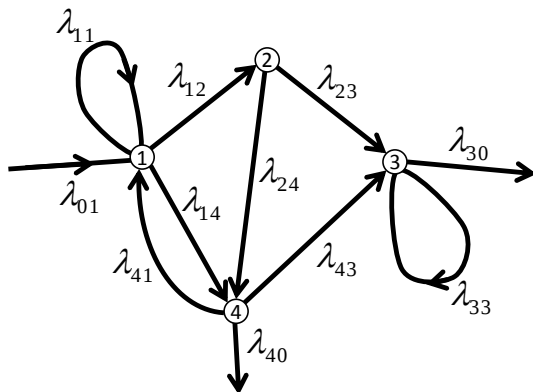
March 26, 2010

Outline

- 1 Jackson Network = a Non-equilibrium Stat Mech
 - Definitions. Product Form for Distribution of Queues.
 - Master Equation. Doi-Peliti technique.
- 2 Statistics of Currents in Jackson Network
 - Non-Equilibrium Phase Transition
 - Example: Station with a Feedback
 - Conclusions and Path Forward

Definitions: Network of $M/M/m/\infty$ queues

Open Jackson (Queuing) Network



- Particles are identical (single class)
- Out of Detailed Balance
- $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) =$
(set of vertexes, set of edges)
- directed graph
- self-loops are allowed
- $M/M/m_i/\infty$ or shorter
 $M/M/m_i$
- Markovian arrival
- Markovian departure
- m_i servers at the node i
- ∞ - waiting room capacity

© Original Artist

Reproduction rights obtainable from
www.CartoonStock.com



"Say-you don't suppose she LIVES in there?"



Agner Krarup Erlang (1878-1929)
Danish engineer, who worked for the
Copenhagen Telephone Exchange,
published the first paper on queueing
theory in 1909.

Queuing Theory Classics

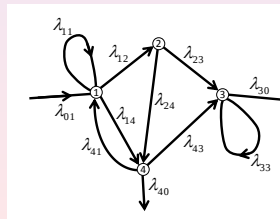
[Kelly '76]

Explicit Solution for Steady Queues over ANY graph!

$$P(\mathbf{n}) = Z^{-1} \prod_{i \in \mathcal{G}_0} \frac{h_i^{n_i}}{\prod_{l_i=1}^{n_i} \theta_{m_i}(l_i)}, \quad \theta_m(l) = \min(m, l)$$

$$\forall i \in \mathcal{G}_0 : - h_i \sum_{j \neq 0}^{(i,j) \in \mathcal{G}_1} \lambda_{ij} + \sum_{j \neq 0}^{(j,i) \in \mathcal{G}_1} \lambda_{ji} h_j + \lambda_{0i} - \lambda_{i0} h_i = 0$$

- Product Form .. but not a Gibbs distribution!
- Stability: $h_i < m_i$
- $\mathbf{h}$ is a “single-particle” object
- It is a version of the “zero range process” model in math/physics [Spitzer '70]
- ... not the story of the Asymmetric Exclusion Process



Master Equation

Many-particle Interacting System

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathbf{n}; t) = & \sum_{(i,j) \in \mathcal{G}_1} \lambda_{ij} \left(\theta_{m_i}(n_i + 1) P(\dots, n_i + 1, \dots, n_j - 1, \dots; t) \right. \\ & \left. - \theta_{m_i}(n_i) P(\dots, n_i, \dots, n_j, \dots; t) \right) \\ & + \sum_{i \in \mathcal{G}_0} \lambda_{0i} (P(\dots, n_i - 1, \dots; t) - P(\dots, n_i, \dots; t)) \\ & + \sum_{i \in \mathcal{G}_0} \lambda_{i0} (\theta_{m_i}(n_i + 1) P(\dots, n_i + 1, \dots; t) - \theta_{m_i}(n_i) P(\dots, n_i, \dots; t)) \\ \theta_m(n) = & \min(n, m). \end{aligned}$$

And still the system allows explicit solution for the stationary distribution !

Doi-Peliti Technique

Allows interpretation as an auxiliary Quantum Mechanics

$$\hat{H} = \sum_{(i,j) \in \mathcal{G}_1}^{i,j \neq 0} \lambda_{ij} (\hat{a}_j^+ - \hat{a}_i^+) \hat{b}_i^{(m_i)} + \sum_{i \in \mathcal{G}_0} \left(\lambda_{0i} (\hat{a}_i^+ - 1) + \lambda_{i0} (1 - \hat{a}_i^+) \hat{b}_i^{(m_i)} \right)$$

$$\hat{b}^{(m)} |n\rangle = \theta_m(n) |n-1\rangle.$$

Peliti ['85] for death-birth + extended to the Q-networks by Zeitak ['07]

$$|s\rangle = \sum_{\mathbf{n}} P(\mathbf{n}) |\mathbf{n}\rangle, \quad \sum_{\mathbf{n}} P(\mathbf{n}) = 1,$$

$$\hat{a}_j^+ |\cdots, n_j, \cdots\rangle = |\cdots, n_j + 1, \cdots\rangle$$

$$\hat{a}_j |\cdots, n_j, \cdots\rangle = n_j |\cdots, n_j - 1, \cdots\rangle.$$

Coherent State Solutions

Ground State of the Hamiltonian is constructed explicitly

$$\hat{H}|\text{coh}_m(h)\rangle = 0$$

$$\hat{b}^{(m)}|\text{coh}_m(h)\rangle = h|\text{coh}_m(h)\rangle, \quad |\text{coh}_m(h)\rangle \equiv g_m(h\hat{a}^+)|0\rangle$$

$$g_m(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{\prod_{l=1}^k \theta_m(l)} = \frac{m^m}{m!} \frac{1}{1-x/m} + \sum_{k=0}^{m-1} x^k \left(\frac{1}{k!} - \frac{m^{m-k}}{m!} \right)$$

$$\forall i \in \mathcal{G}_0 : -h_i \sum_{\substack{(i,j) \in \mathcal{G}_1 \\ j \neq 0}} \lambda_{ij} + \sum_{\substack{(j,i) \in \mathcal{G}_1 \\ j \neq 0}} \lambda_{ji} h_j + \lambda_{0i} - \lambda_{i0} h_i = 0$$

- Related Operator Technique (and results !!) by Massey ['84]

Notice: the entire spectrum does not allow explicit expression!

Outline

- 1 Jackson Network = a Non-equilibrium Stat Mech
 - Definitions. Product Form for Distribution of Queues.
 - Master Equation. Doi-Peliti technique.
- 2 Statistics of Currents in Jackson Network
 - Non-Equilibrium Phase Transition
 - Example: Station with a Feedback
 - Conclusions and Path Forward

Joint Distribution function of queues and currents

- Assume that the system is stable: $\forall i : h_i < m_i$
- Currents are defined on edges: J_{ij} is the number of particles which traversed the edge in time t

Our study:

- Joint probability distribution function of queues and currents accumulated in time t : $P(\mathbf{n}, \mathbf{J}|t)$
- When t is sufficiently large (larger than correlation time) we are in the Large Deviation regime:
$$P(\mathbf{n}, \mathbf{J}|t) \sim \exp(-tS(\mathbf{J}/t))\Phi(\mathbf{n}; \mathbf{J}/t)$$
- How universal is the Crámer function, $S(\mathbf{J}/t)$?
- How does $\Phi(\mathbf{n}; \mathbf{J}/t)$ change with increase in $\mathbf{J}$? Anomaly?

Non-Congested to Congested (II)

$$P(\mathbf{n}, \mathbf{J}|t) \sim \exp(-tS(\mathbf{J}/t))\Phi(\mathbf{n}; \mathbf{J}/t)$$

Our statements (II)

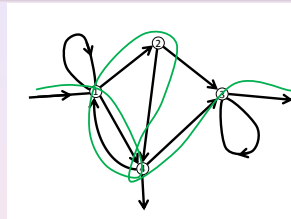
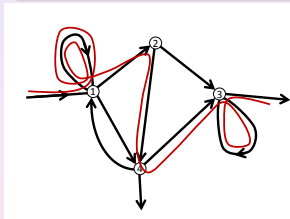
- In the **congested** regime one or more nodes are **congested** = all respective moments of $\mathbf{n}$ grow (to become infinite)
- The transition is **second order** = $1/\langle n_i \rangle$ is the order parameter, $\rightarrow 0$ at the transition point

► Technical Details

- Extension of the Doi-Peliti technique for generation function (over currents)
- Explicit construction of the effective Hamiltonian ground state in the coherent state form

Universality of the uncongested regime

Consider a trajectory (red or green)



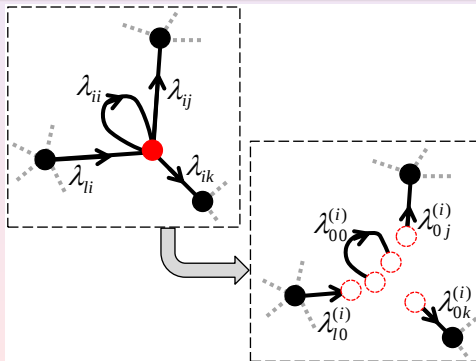
Statistics of currents is interaction independent !

- Statistics of trajectories (and of the passing time!) is not sensitive (in the leading order) to interaction
- Statistics of passing time is interaction sensitive
- # of particles which went through the system in a large time is much larger than # of these which did not leave the system

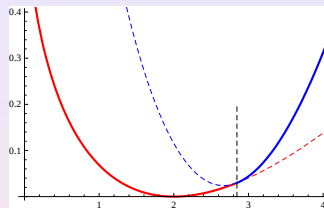
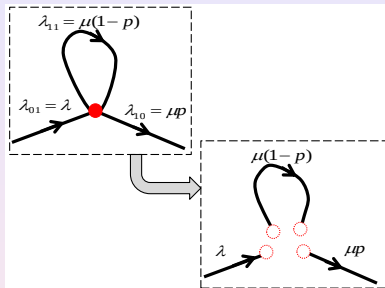
Congested regime is as simple to describe

Graph Reduction Procedure

in the congested regime reduces the problem to the previous one on a less connected network



Station with a Feedback: Uncongested to Congested

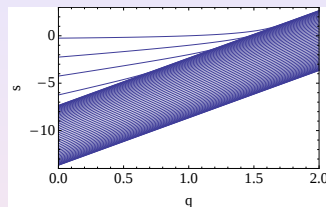
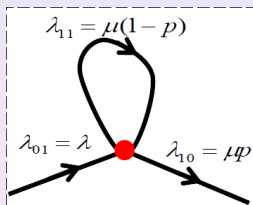


$$j < j_c : \quad S(j) = \lambda(1 - p/2) - \frac{\sqrt{p\lambda(4j + p\lambda)}}{2} + j \ln \left(\frac{2j + p\lambda - \sqrt{p\lambda(4j + p\lambda)}}{2j(1 - p)} \right) .$$

$$j > j_c : \quad S(j) = \frac{\lambda^2 + m^2 p \mu^2 - 2\lambda m p \mu - m^2 p(1 - p)\mu^2 \ln(\mu p m / \lambda)}{\lambda} + j \ln \left(\frac{j}{em\mu(1 - p)} \right)$$

- The transition is of the second order = jump in the Crámer function derivative

Numerical Exploration of the Spectrum



$$m = 5, \lambda = 0.5, \mu = 2, p = 0.5, N = 50$$

- At $q = 1$ $\Delta = 0$. Three discrete states and continuous band
- Gap is closing at q_c (variable conjugated to j_c)
- Supported by analytical analysis
- Consistent with the general (ground state) picture

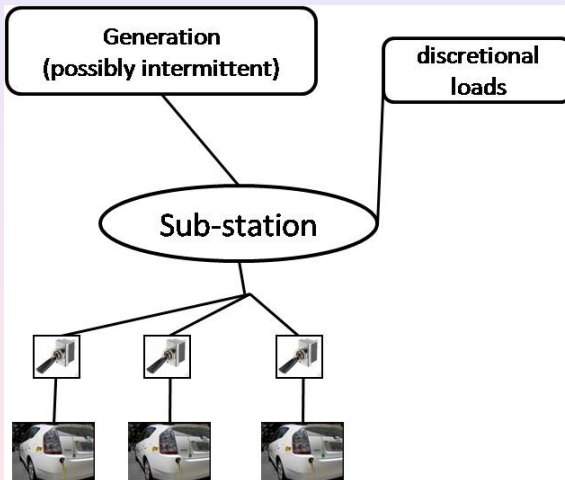
Conclusions

- Studied Generating function of currents over Jackson (queuing) network
- Adopted Doi-Peliti (Massey) technique
- Explicit coherent state (product form) solution for the ground state
- Observed second order dynamical transition from uncongested to congested network
- Illustrated and validated the story on example of a single station with feedback

Path Forward

- Sequence of phase transitions. Algorithmically and Analytically.
- Random Queuing Network - thermodynamic limit.
- Generalization to rates dependent on the outgoing (and much more difficult incoming or all) queues
- Generalization to multi-class (priorities)
- Finite waiting room - makes the problem really difficult (truly multi-particle, quantum)
- Extension to semi-open and open networks
- Changing with time rates - fluctuation theorem and Jarzynski relations?
- Some rigorous proofs are required (would not hurt)
- Applications to optimization and control, e.g. in energy and more specifically **power grid networks**

An application of interest



PHEV charging, power and traffic control



Thank you !

Extension of the operator (Doi-Peliti) technique

$$\partial_t |\mathbf{s}(\mathbf{n}; \mathbf{J})\rangle = \left(\hat{H} + \sum_{(i,j) \in \mathcal{G}_1} \hat{J}_{ij} \right) |\mathbf{s}(\mathbf{n}; \mathbf{J})\rangle,$$

$$\forall i, j \neq 0: \quad \hat{J}_{ij} = \lambda_{ij} (\hat{a}_{ij}^+ - 1) \hat{a}_j^+ \hat{b}_i^{(m_i)},$$

$$\hat{J}_{0i} = \lambda_{0i} (1 - \hat{a}_{0i}^+) \hat{a}_i^+, \quad \hat{J}_{i0} = \lambda_{i0} (\hat{a}_{i0}^+ - 1) \hat{b}_i^{(m_i)}$$

$$|\mathbf{s}(\mathbf{n}; \mathbf{J})\rangle = P(\mathbf{n}; \mathbf{J}) |\mathbf{n}; \mathbf{J}\rangle. \quad |\mathbf{s}_{\mathbf{q}}(\mathbf{n})\rangle = \sum_{\mathbf{J}} \prod_{(i,j) \in \mathcal{G}_1} q_{ij}^{J_{ij}} |\mathbf{s}(\mathbf{n}; \mathbf{J})\rangle$$

$$\partial_t |\mathbf{s}_{\mathbf{q}}(\mathbf{n})\rangle = \hat{H}_{\mathbf{q}} |\mathbf{s}_{\mathbf{q}}(\mathbf{n})\rangle,$$

$$\begin{aligned} \hat{H}_{\mathbf{q}} = & \sum_{(i,j) \in \mathcal{G}_1} \lambda_{ij} (\hat{a}_j^+ - \hat{a}_i^+) \hat{b}_i^{(m_i)} + \sum_{(i,j) \in \mathcal{G}_1}^{i \neq 0, j \neq 0} \lambda_{ij} (q_{ij} - 1) \hat{a}_j^+ \hat{b}_i^{(m_i)} \\ & + \sum_{(0,i) \in \mathcal{G}_1} \lambda_{0i} (q_{0i} a_i^+ - 1) + \sum_{(i,0) \in \mathcal{G}_1} \lambda_{i0} (q_{i0} - a_i^+) \hat{b}_i^{(m_i)}. \end{aligned}$$

