A long wave electrodynamic model of an array of small Josephson junctions

Frontiers in nonlinear waves, Tucson, March 26-29 2010

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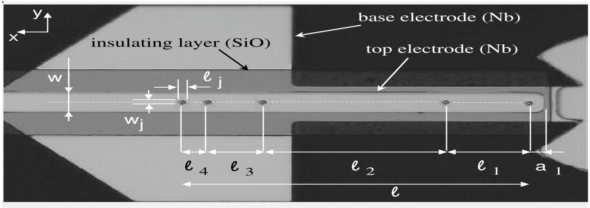
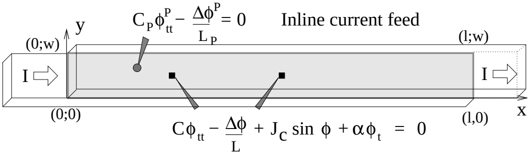
Parallel array of small Josephson junctions (Microwave mixer)

F. Boussaha and M. Salez, Observatoire de Paris
$l = 100\mu m, w = 7\mu m,\ l_j = 1\mu m,\ w_j = 1\mu m.$



- Interaction between junction and microstrip
- Coupling between junctions
- Design of devices

Resistively Shunted Junction Model



$$C\Phi_{tt} - \nabla\left(\frac{1}{L}\nabla\Phi\right) + J_c \sin\left(\frac{\Phi}{\Phi_0}\right) + \frac{\Phi_t}{R} = 0.$$

$L = \mu_0 d_0$ branch inductance, C capacity / surface, J_c critical current density,

$\alpha = 1/R$ quasiparticles, $\Phi_0 = \frac{\hbar}{2e}$

$\nu = 0$ (1) inline (overlap)

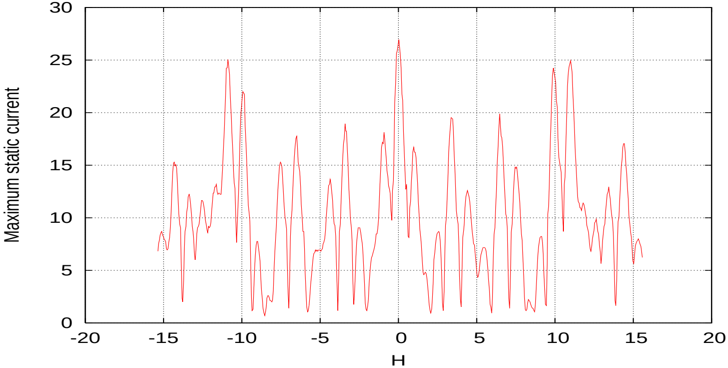
Boundary conditions

$\Phi_x|_{x\in\{0;l\}} = d_0 H \mp (1-\nu)\frac{L}{2w} I \qquad \Phi_y|_{y\in\{0:w\}} = \mp\nu\frac{L}{2l} I$

Assume L constant

Maximum current for static solution : experiment

M. Salez et al, J. of Appl. Physics (2007)

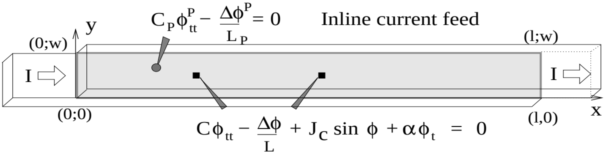


A device : F. Boussaha and M. Salez

Device	w	a_1	a_2	a_3	a_4	a_5
A	5	2	22	64	76	82

in μm, junctions 1×1

Normalized 2D model



$$\varphi_{tt} - \Delta\varphi + g(x,y)[\kappa\varphi_{tt} + \sin(\varphi) + \alpha\varphi_t] = 0$$

Units of length, time and flux $\lambda_J = \sqrt{\frac{\Phi_0}{J_c L}}, \quad \frac{1}{\omega_p} = \sqrt{\frac{C_P\Phi_0}{J_c}}$

$g = 1, 0, \quad , \kappa = \frac{C}{C_P} - 1, \quad \alpha = \frac{1}{R}\sqrt{\frac{\Phi_0}{J_c C_P}}.$

Boundary conditions

$\varphi_x|_{x=0,l} = H \mp (1-\nu)\frac{I}{2w}, \quad \varphi_y|_{y=0,w} = \mp\nu\frac{I}{2l}$

What to do ? Global analysis, 2D numerics, Reduced model

Outline of talk

1D continuous model

Expand φ on transverse Fourier modes

$$\varphi(x,y,t) = \frac{\nu I}{2wl}\left(y - \frac{w}{2}\right)^2 + \sum_{n=0}^{\infty} \phi_n(x,t)\cos\left(\frac{n\pi y}{w}\right) \ ,$$

0 mode

$$\phi_{tt} - \phi_{xx} + \frac{w_j}{w} g(x, y=0)(\kappa\phi_{tt} + \alpha\phi_t + \sin\phi) = \nu\frac{\gamma}{l}$$

$$\gamma = \frac{I}{w}$$

w_j/w "rescaling" of $\lambda_J(=1)$ into $\lambda_{eff} = \sqrt{\frac{w}{w_j}} > 1$

JGC et al, J. of Modern Physics C, vol. 7, No. 2, 191-216, (1996).

1D continuous model and 1D continuous/discrete model

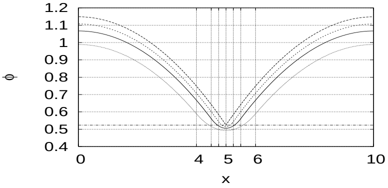
$$\phi_{tt} - \phi_{xx} + g_d\left[\sin(\phi) + \alpha\phi_t\right] = \gamma/l\ .$$

$$g_d(x) = \begin{cases} w_j^2/(dw) & |x - a_i| \le d/2 \\ 0 & \text{elsewhere} \end{cases}$$

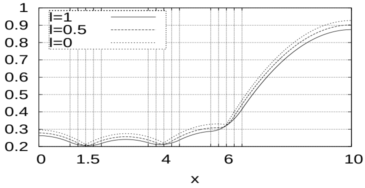
Total current of junction constant : $\int_0^l g_d\, dx = w_j^2/w$.

ϕ with different d, $\gamma = 0.5$, static regime.

1 Josephson junction, l = 2,1,0.5,0



With 3 Josephson junctions.



$$\lim_{d\to 0}\int_{a-d/2}^{a+d/2} \phi_{tt} - \phi_{xx} + g_d(\sin(\phi) + \alpha\phi_t)dx \ = -[\phi_x]_{a-}^{a+} + \frac{w_j^2}{w}(\sin(\phi|_a) + \alpha\phi_t|_a)$$

1D continuous/discrete model

n junctions device

$$\phi_{tt} - \phi_{xx} + \sum_{i=1}^{n} d_i \delta(x - a_i) \left[\kappa \phi_{tt} + \sin(\phi) + \alpha \phi_t\right] = \nu \frac{\gamma}{l} ,$$

with, $\begin{cases} \phi_x(0;t) = H - (1-\nu)\gamma/2, \\ \phi_x(l;t) = H + (1-\nu)\gamma/2. \end{cases}$ and $d_i = w_i^2/w$.

Advantages :

1. Evolution of ϕ is not neglected in the linear part.
2. Interaction between the junction and microstrip is considered in time **and in space**.
3. Approach remains coherent when we change, independently, the position **and area** of each junctions.
4. Non uniform devices, $0 - \pi$ junctions..

Maximum static current

Maximal γ for H given.

$$-\phi_{xx} + \sum_{i=1}^{n} d_i \delta(x - a_i) \sin(\phi) = \nu \frac{\gamma}{l} \, ,$$

with, $\begin{cases} \phi'(0;t) &= H - (1-\nu)\gamma/2, \\ \phi'(l;t) &= H + (1-\nu)\gamma/2. \end{cases}$

$\sum_{i=1}^{n} d_i \sin(\phi_i) = \gamma$

J. G. C. and L. Loukitch, SIAM J. of Appl. Math. (2007)

Numerical methods

shooting method : Polynomial by part depends on $\phi(a_1)$ and γ (H, ν fixed)

$P_1(x) = -\frac{\nu\gamma}{2l}(x^2 - a_1^2) + \left(H - (1-\nu)\frac{\gamma}{2}\right)(x - a_1) + \phi(a_1)$

$P_{k+1}(x) = P_1(x) + \sum_{m=1}^{k} d_m \sin(P_m(a_m))(x - a_m)$

Boundary value problem, nonlinear system : $n = 5$

$\phi = \psi - \nu\frac{\gamma}{l}\frac{x^2}{2} \equiv \psi - f(x)$

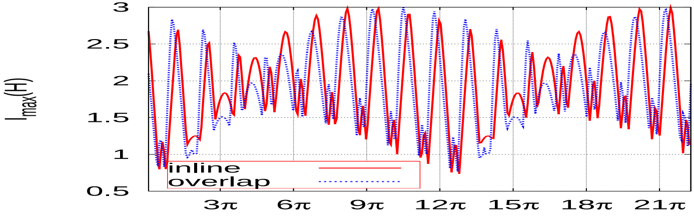
$$
\begin{aligned}
H - (1-\nu)\frac{\gamma}{2} - \frac{\psi_2 - \psi_1}{a_2 - a_1} + d_1 \sin(\psi_1 - f_1) &= 0,\\
-\frac{\psi_3 - \psi_2}{a_3 - a_2} + \frac{\psi_2 - \psi_1}{a_2 - a_1} + d_2 \sin(\psi_2 - f_2) &= 0,\\
-\frac{\psi_4 - \psi_3}{a_4 - a_3} + \frac{\psi_3 - \psi_2}{a_3 - a_2} + d_3 \sin(\psi_3 - f_3) &= 0,\\
-\frac{\psi_5 - \psi_4}{a_5 - a_4} + \frac{\psi_4 - \psi_3}{a_4 - a_3} + d_4 \sin(\psi_4 - f_4) &= 0,\\
-H - (1+\nu)\frac{\gamma}{2} + \frac{\psi_5 - \psi_4}{a_5 - a_4} + d_5 \sin(\psi_5 - f_5) &= 0.
\end{aligned}
$$

The Periodicity $\gamma_{max}(H) = \gamma_{max}(H + H_p)$

$l_j = a_{j+1} - a_j$, and $l_{min} = \min_j(l_j)$

- **Harmonic array** : If l_i is a multiple of l_{min} for all i then $H_p = \frac{2\pi}{l_{min}}$

- **General case**, if $l_j = p_j/q_j$, p_j and q_j are integers, then $H_p = 2\pi \frac{LCM(q_1;...;q_{n-1})}{HCF(p_1;...;p_{n-1})}$.



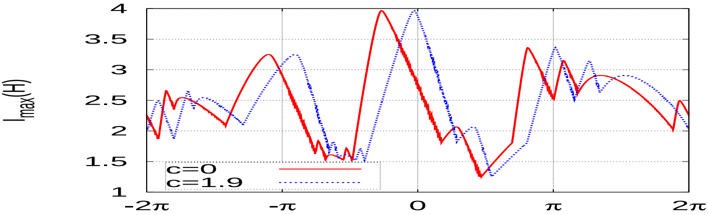
$a_1 = 1$, $a_2 = 1 + \frac{3}{2}$ and $a_3 = 1 + \frac{3}{2} + \frac{5}{3}$.

$H_p = 2\pi LCM(2;3)/HCF(3;5) = 12\pi$.

Magnetic shift

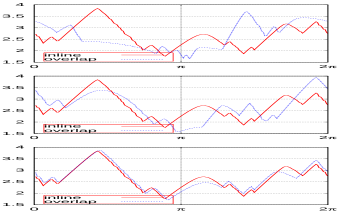
C1 $\{a_1, \ldots, a_n\}$, C2 $\{a_1 + c, \ldots, a_n + c\}$, same l, ν, d_i

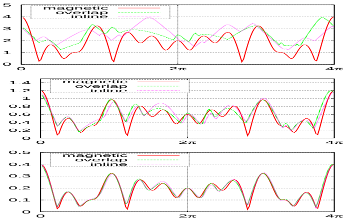
$\gamma_{\max}(H)|_{C2} = \gamma_{\max}(H + H_\nu)|_{C1}$ avec $H_\nu = \nu\gamma(a_n + a_1 - l)/(2l)$



$l = 10$, $\nu = 1$, $a_2 - a_1 = 1.5$, $a_3 - a_2 = 2.5$, $a_4 - a_3 = 2$

Effects of ν, l et d_j : convergence





$l = 8, 16$ et 64 $\qquad$ $d_j = 1, 0.3$ et 0.1.

$\gamma_{\max}(H)$ with $\nu \neq 0 \rightarrow \gamma_{\max}(H)$ with $\nu = 0$ (same circuit) when $\nu \left(\sum_j d_j \right) / l \rightarrow 0$

Inline and overlap $\rightarrow$ magnetic approximation for $d_j \rightarrow 0$

Magnetic approximation,

$d_j \to 0$, $\phi(x) \to Hx + C$, for $d_j \ll 1$

$$\gamma(H) = \sum_{j=1}^{N} d_j \sin(Ha_j + C),$$

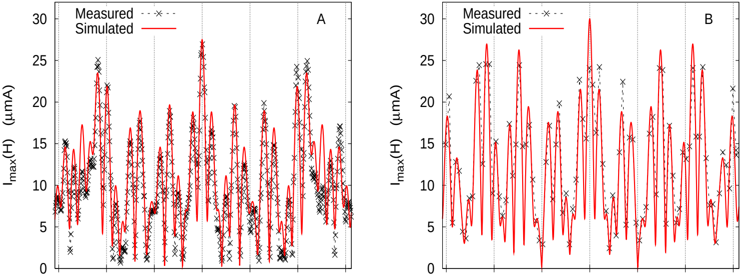
$$c_{max}(H) = \arctan\left(\frac{\sum_{i=1}^{N} d_j \cos(Ha_j)}{\sum_{j=1}^{N} d_j \sin(Ha_j)}\right),$$

$$\gamma_{max}(H) = \left| \sum_{j=1}^{N} d_j \sin\left(Ha_j + c_{max}(H)\right) \right|$$

J. H. Miller et al, Appl. Phys. Lett. 59, 3330, (1991).

Magnetic approximation, experiments

M. Salez et al, J. of Appl. Physics (2007)



A and B devices : F. Boussaha and M. Salez

Device	w	a_1	a_2	a_3	a_4	a_5
A	5	2	22	64	76	82
B	7	2	20	64	75	82

in μm, junctions 1×1

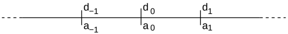
Symetric device :

Symetric device : $a_j = -a_{-j}$ et $d_j = d_{-j}$,
For junction j position a_j and strength $d_j = w_j^2/w$

Properties

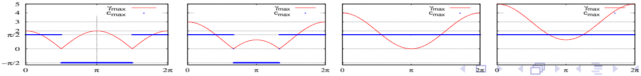
1. Moving junction unit does not change $\gamma_{\max}(H)$
2. $\gamma_{\max}(H) = \gamma_{\max}(-H)$
3. Current maximum for $c_{\max}(H) = \pm\pi/2$,
   $\gamma_{max}(H) = |d_0 + 2\sum_{j=1}^{(n+1)/2} d_j \cos(Ha_j)|$

Three junctions : $I_{max}(H) = |d_0 + 2d_1 \cos(Ha_j)|$



$$a_{-1} = -1,\ a_0 = 0,\ a_1 = 1,\ d_{-1} = d_1 = 1$$

$d_0 = 0$ $d_0 = 1$ $d_0 = 2$ $d_0 = 3$ (Likharev)



Inverse problem : position and sizes of junctions from $\gamma_{\max}(H)$

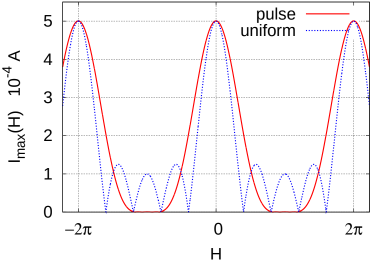
J.G.C. and L. Loukitch, Inverse Problems, 2008

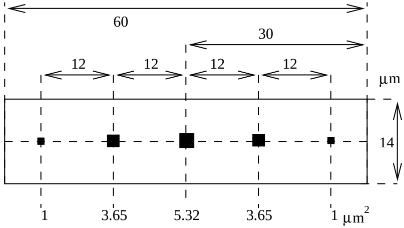
Proposition

Assume a $\gamma_g(H)$ even, periodic of period H_p and strictly positive. The array is harmonic and the positions of the junctions are given by $a_i = i2\pi/H_p$ where i is an integer. Their strengths d_i are given by

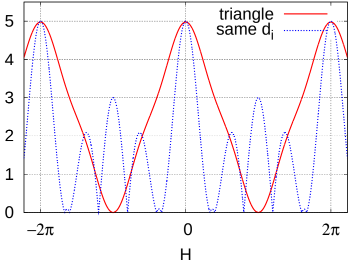
$$d_{-i} = d_i = \frac{1}{H_p}\int_0^{H_p} \gamma_g(H)\cos(Ha_i)dH \ , \quad \forall i \in \{0, \ldots, n/2\} \ . \quad (1)$$

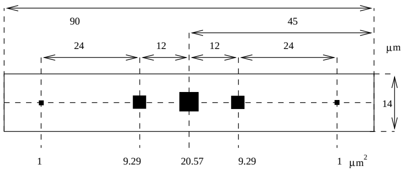
Equidistant junctions



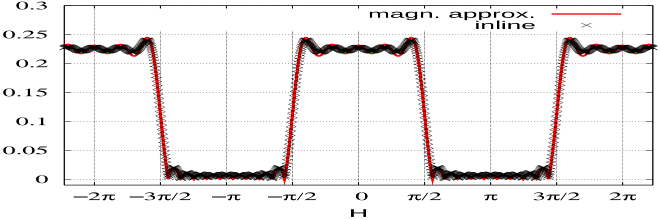


Non equidistant junctions





Inverse problem : square $\gamma_{\max}(H)$



i	0	1	2	3	4	...
a_i	0	1	2	3	4	...
$2\pi d_i$	$2h$	0	$-\frac{\sin(2\pi h)}{\pi}$	0	$\frac{\sin(4\pi h)}{2\pi}$	...

High voltage solution

Rotating phase approximation $\phi \approx Vt + \phi_v(x)$

$-\phi_{v\,xx} + \alpha V \sum_{j=1}^{n} d_j \delta(x - a_j) = \nu \frac{\gamma}{l}$

$\phi_{v\,x}|_{x=0} = H - (1-\nu)\frac{\gamma}{2}, \quad \phi_{v\,x}|_{x=l} = H + (1-\nu)\frac{\gamma}{2}$

Solution exists if $V = \frac{\gamma}{\alpha \sum_j d_j}$

Simplify problem $\psi = \phi - \phi_v$

$\psi_{tt} - \psi_{xx} + \sum_{j=1}^{n} d_j \delta(x - a_j) \left[\kappa \psi_{tt} + \alpha \psi_t + \sin(\psi + \Psi_j) - \frac{\gamma}{\sum_{k=1}^{n} d_k} \right] = 0$

Boundary conditions $\psi_x|_{x=0} = 0, \quad \psi_x|_{x=l} = 0$

Adapted spectral problem

$$\psi_{tt} - \psi_{xx} + \sum_{j=1}^{n} d_j \delta(x - a_j)\kappa\psi_{tt} = 0$$

BCs $\psi_x|_{0,l} = 0$

$$\psi(x,t) = e^{i\omega t}\varphi(x)$$

$$\varphi_{xx} + \omega^2 \left[1 + \sum_{j=1}^{n} d_j \delta(x - a_j)\kappa\right] \varphi = 0,$$

$\kappa = 0 \rightarrow$ standard (harmonic) cosine Fourier modes

$\kappa \neq 0 \rightarrow$ new modes, non harmonic

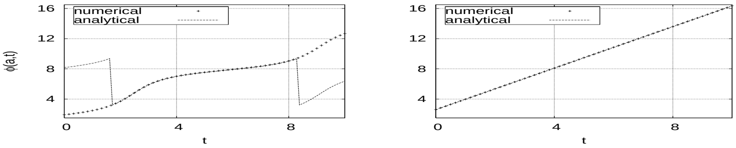
$\kappa = 0$, single junction in cavity

$$lA_0'' + d_1\left(\alpha\psi_{1t} + \sin(\psi_1) - \frac{\gamma}{d_1}\right) = 0\ ,$$

$$A_p'' + \left(\frac{p\pi}{l}\right)^2 A_p + \frac{2d_1}{l} c_p^1 \left(\alpha\psi_{1t} + \sin(\psi_1) - \frac{\gamma}{d_1}\right) = 0\ ,$$

ohmic mode : cavity driven by junction,
$\alpha\phi_t|_a - \gamma/d_a = 0$, $\quad [\phi_x]_{a-}^{a+} = d_a \sin(\phi|_a) - \gamma$.

junction mode : ϕ behaves the same in the whole circuit,
$\sin(\phi|_a) + \alpha\phi_t|_a - \gamma/d_a = 0$, $\quad [\phi_x]_{a-}^{a+} = -\gamma$.



$\phi(a,t)$: junction mode ohmic mode

Dissipative kink solution $\kappa, H \ll 1$

One junction only

$$\psi_{tt} - \psi_{xx} + d_1 \delta(x - a_1) \left[\alpha \psi_t + \sin(\psi) - \frac{\gamma}{d_1} \right] = 0 \ ,$$

Solution of wave equation $\quad \psi(x; t) = f(x \pm t) \equiv f(\xi)$

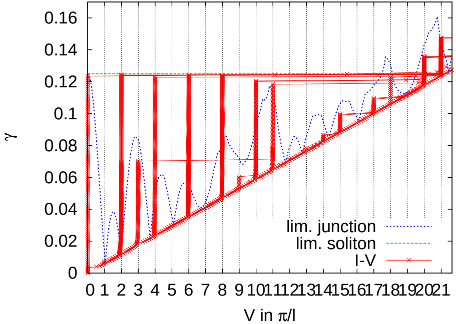
Solution in junction $\quad \alpha f' + \sin(f) - \frac{\gamma}{2d_1} = 0$

$$\psi(x,t) = \begin{cases} 2\arctan\left(1 - 2\alpha/(x \pm t + C)\right) + 2k\pi \ , & \text{for } x \pm t < C \ , \\ \pi/2 \ , & \text{for } x \pm t = C \ , \quad k \in \\ 2\arctan\left(1 - 2\alpha/(x \pm t + C)\right) + 2(k+1)\pi \ , & \text{for } x \pm t > C \ , \end{cases}$$
$\mathcal{N}$.

Velocity $=1$, width increases with α

Kinks can exist for a few junctions if $\Psi_j \ll 1$

$I - V$ curves of the 5 junction device, $\kappa = 0$

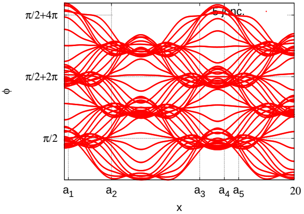


$I - V$ curves of the device of the top photo, 5 junctions, $d_i = 0.025$, $\kappa = 0$, $\alpha = 0.3$, $H = 0$, $\nu = 0$.

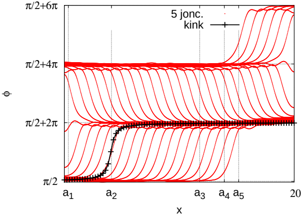
lim. soliton$(V) = I_{\max}(H) = 0.125$

lim. junction$(V) = \left|\frac{\sum_{j=1}^{n} d_j \cos(V a_j)}{\sup_{k\in\{1,\dots,n\}} d_k \cos(V a_k)}\right| \left[\sqrt{(\alpha V)^2 + 1} - \alpha V\right] + \alpha V \sum_{i=1}^{n} d_i$

$\phi(x;t)$, $x \in [0;l]$, $t = 0.8k$, $k \in \mathbb{N}$



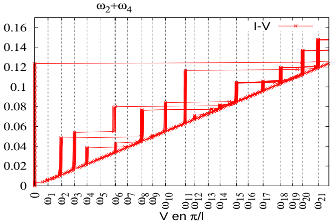
lim. junction : $V = 3\pi/l$, $\gamma = 0.07$



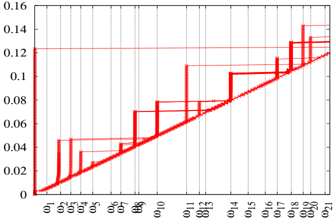
lim. soliton : $V = 2\pi/l$, $\gamma = 0.125$

Same device but $\kappa \geq 0$

$\kappa = 16$:



$\kappa = 64$:



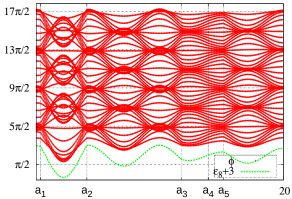
Associated linear problem ;

$$\phi_{tt} - \phi_{xx} + \sum_{i=1}^{n} d_j \delta(x - a_j)\kappa\phi_{tt} = 0.$$

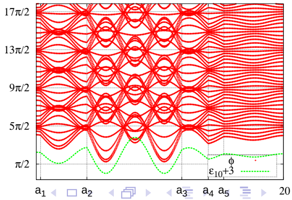
ω_n eigenvalues : position of resonances.

$\kappa = 64$, ϕ vs ε_n

$\phi(x;t)$, $x \in [0;l]$, $t = 0.4k$, $k \in \mathbb{N}$



$V = \omega_8 \approx 1.086$, $\gamma = 0.0724$



Conclusion

Statics well understood

Dynamics

- **Localized sine nonlinearities (junctions) excite the whole spectrum of a cavity**
- Small capacity miss-match ($\kappa \approx 0$)
  IV curve well understood (resonances, kinks)
- Large capacity miss-match ($\kappa \gg 0$)
  Linear problem dominates, new eigenfrequencies and eigenmodes