Isochronous dynamical systems, the arrow of time and the definitions of “chaotic” versus “integrable” behaviors

Francesco Calogero
Physics Department, University Roma “La Sapienza”, 00185 Rome (Italy)
Istituto Nazionale di Fisica Nucleare, Sezione di Roma

Summary

Any (autonomous) dynamical system can be *extended* or *modified*, obtaining thereby a *new* (autonomous) dynamical system involving a constant *T*---the value of which can be freely assigned---and featuring the following two properties: (i) all solutions of the new model are *isochronous* (completely periodic in all their degrees of freedom with the assigned period *T*); (ii) starting from *generic* initial data, the time evolution of the new dynamical system over time intervals of order $\tilde{T} << T$ is *essentially identical* to that of the original dynamical system, up to a *constant* rescaling of time and of corrections of order $\tilde{T}/T$. These findings entail that, in some sense, “*isochronous systems are not rare*” and moreover that such systems may feature an “*extremely complicated* “ time-evolution. They are also valid in the context of *Hamiltonian* dynamics; they are in particular applicable to the most general many-body problem (provided it is, overall, translation-invariant), entailing remarkable observations about statistical mechanics, thermodynamics and the issue of the “arrow of time” for macroscopic physics. Since *completely periodic* systems are *maximally superintegrable* (possessing the maximal number of functionally independent constants of motion compatible with the time evolution not being frozen), these findings also entail that *any* (Hamiltonian) dynamics can be *embedded* into a *superintegrable* (Hamiltonian) dynamics; and again, that such *superintegrable* systems may feature an “*extremely complicated* “ time-evolution.

All these findings have been obtained together with **François Leyvraz**. Some of them are reported in a recent monograph (F. Calogero, *Isochronous systems*, Oxford University Press, 2008); others are more recent, see references listed below. A very recent finding demonstrates how to extend an *arbitrary* (autonomous) dynamical system so that the (also autonomous) extended system is *isochronous* (with an *arbitrarily assigned* period *T*) yet its dynamics for an *arbitrary* fraction (of course, less than unity) of its (periodic) time evolution is *exactly identical* to that of the original system.

These findings suggest the need to invent new definitions **associated with a *finite* time scale** of “chaotic” versus “integrable” behaviors of dynamical systems (all current definitions refer instead to the behavior over *infinite* time).

Main references

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A trick to transform a dynamical system into an *isochronous* dynamical system

Let us start from an *autonomous*, but otherwise largely arbitrary, dynamical system, say

$$\underline{X}' = \underline{h}\left(\underline{X}\right); \quad X_n' = h_n\left(\underline{X}\right), \quad n = 1, \dots, N, \quad (*)$$

with the appended prime indicating differentiation with respect to the variable τ of $\underline{X}(\tau)$.

We then *change* it so that it reads as follows:

$$\underline{\dot{x}} = \dot{\tau}(t)\,\underline{h}\left(\underline{x}\right); \quad \dot{x}_n = \dot{\tau}(t)\,h_n\left(\underline{x}\right), \quad n = 1, \dots, N, \quad (**)$$

with the superimposed dot indicating differentiation with respect to the variable t of $\underline{x}(t)$ and $\tau(t)$.

It is then plain that the general solution of this system reads

$$\underline{x}(t) = \underline{X}\left[\tau(t)\right]; \quad x_n(t) = X_n\left[\tau(t)\right], \quad n = 1, \dots, N,$$

where $\underline{X}(\tau)$ is the general solution of (*). Hence if the scalar function $\tau(t)$ is periodic with period T,

$$\tau(t + T) = \tau(t),$$

the general solution of (**) *inherits* from $\tau(t)$ the property to be *isochronous* with period T:

$$\underline{x}(t + T) = \underline{x}(t).$$

In conclusion, whenever $\underline{h}\left(\underline{X}\right)$ is such that the solution of the system (*) of N nonlinear ODEs exists globally, *all* solutions of the system (**) are periodic with period T: this system is ***isochronous***.

However, the system (**) is *not* autonomous. To eliminate this "defect", we perform the second step of our treatment, replacing this system (**) with the system

$$\dot{\underline{x}} = \phi\, \underline{h}\left(\underline{x}\right);\ \ \dot{x}_n = \phi\, h_n\left(\underline{x}\right),\ \ n = 1,\ldots, N,\ \ (***)$$

which is of course equivalent provided the time-evolution of the scalar quantity ϕ is such that

$$\phi(t) = \dot{\tau}(t).$$

There are now two options to obtain a quantity ϕ that qualifies for this purpose. One option---perhaps the most obvious---is to treat ϕ as an additional dependent variable, and to *extend* the system (*) by attaching to it a few additional ODEs involving ϕ and possibly other, additional dependent variables, so as to guarantee that the time evolution of ϕ has the desired property. Specific instances of how to achieve this goal are detailed in a paper [F. Calogero and F. Leyvraz, “How to extend any dynamical system so that it becomes isochronous, asymptotically isochronous or multi-periodic”, J. Nonlinear Math. Phys. **16**, 311-338 (2009)], where it is also shown how a third step of our treatment---consisting essentially in a change of the additional dependent variables, entangling them with the original variables---allows to manufacture many, quite neat, extended dynamical systems having the required properties of *isochrony*. One example obtained in this manner is displayed below.

Another option is to identify a collective variable $\phi(\underline{x})$ that, as a consequence of the very evolution entailed by the dynamical system (***), has a time evolution, $\phi(t) \equiv \phi[\underline{x}(t)]$, such that, via the preceding formula, it defines a function $\tau(t)$ having the desired properties. This is the approach on which we will then focus, restricting moreover attention to a Hamiltonian system of major physical relevance.

Example: a modified Lorenz system

$$\dot{x}_1 = -\alpha\, y_1 \left(1 + x_1^2\right)\left(x_1 - x_2\right),$$

$$\dot{x}_2 = y_1 \left(1 + x_1^2\right)\left(\beta\, x_1 - x_2 - x_1\, x_3\right),$$

$$\dot{x}_3 = y_1 \left(1 + x_1^2\right)\left(x_1\, x_2 - \gamma\, x_3\right),$$

$$\dot{y}_1 = \Omega\, y_2 + 2\alpha\, x_1 \left(x_1 - x_2\right) y_1^{\,2},$$

$$\dot{y}_2 = -\Omega\, y_1 + 2\alpha\, x_1 \left(x_1 - x_2\right) y_1\, y_2 \,.$$

For $\Omega = 0$ one can ignore y_2 (whose evolution does not influence the other variables) and set $y_1\left(1 + x_1^2\right) = 1$: then the fourth of these 5 ODEs coincides with the first, and the first 3 become the 3 ODEs of the Lorenz model:

$$X_1{}' = -\alpha\,(X_1 - X_2)\,,$$

$$X_2{}' = \beta\, X_1 - X_2 - X_1 X_3\,,$$

$$X_3{}' = X_1 X_2 - \gamma X_3\,.$$

Here the appended prime indicates differentiation with respect to the independent variable, which for convenience we call τ, so that

$$X_n \equiv X_n(\tau),\quad X_n{}' \equiv X_n{}'(\tau) \equiv dX_n(\tau)/d\tau\,.$$

But in fact also the solution of our *isochronous* model --- with $\Omega > 0$ --- can be “explicitly” exhibited in terms of the solutions $X_n(\tau)$ of the Lorenz model, with the same initial data,

$$X_n(0) = x_n(0)\,:$$

$$x_n(t) = X_n[\tau(t)], \quad n = 1,2,3,$$

$$y_1(t) = \dot{\tau}(t) / \left\{1 + [x_1(t)]^2\right\},$$

$$y_2(t) = \Omega^{-1}\ddot{\tau}(t) / \left\{1 + [x_1(t)]^2\right\},$$

$$\tau(t) = \{A \sin(\Omega t) + B[1 - \cos(\Omega t)]\} / \Omega \ .$$

This solution is clearly periodic with period

$$T = 2\pi / \Omega \ .$$

And clearly, for $0 < t << T$,

$$\tau(t) = At + O(t/T).$$

A trick to *modify* a Hamiltonian into an *isochronous* Hamiltonian

$$\left[\ H\left(\underline{p},\underline{q}\right),\Theta\left(\underline{p},\underline{q}\right)\ \right]=1$$

Isochronous Hamiltonian:

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\frac{1}{2}\left\{\left[H\left(\underline{p},\underline{q}\right)\right]^2+\Omega^2\left[\Theta\left(\underline{p},\underline{q}\right)\right]^2\right\};\quad T=\frac{2\pi}{\Omega}$$

“Isochronous Hamiltonian systems are not rare”

For a proof see: F. Calogero and F. Leyvraz, “General technique to produce isochronous Hamiltonians”, J. Phys. A.: Math. Theor. **40**, 12931-12944 (2007).

A remarkable example: the many-body problem

We write as follows the (simplest version of the) Hamiltonian characterizing the standard nonrelativistic *N*-body problem:

$$H\left(\underline{p},\underline{q}\right)=\frac{1}{2}\sum_{n=1}^{N}\left[p_n^2\right]+V\left(\underline{q}\right),\ V(\underline{q}+a)=V(\underline{q}).$$

Let us now review some standard related developments, trivial as they are.

We hereafter denote with P the total momentum, and with Q the (canonically-conjugate) centre-of-mass coordinate:

$$P=\sum_{n=1}^{N} p_n,\ Q=\frac{1}{N}\sum_{n=1}^{N} q_n .$$

Thanks to the translation invariance property

$$\left[H,P\right]=0$$

Here and hereafter the Poisson bracket $[F,G]$ of two functions $F\left(\underline{p},\underline{q}\right)$ and $G\left(\underline{p},\underline{q}\right)$ of the canonical variables is defined as follows:

$$\left[F,G\right]=\sum_{n=1}^{N}\left[\frac{\partial F\left(\underline{p},\underline{q}\right)}{\partial p_n}\frac{\partial G\left(\underline{p},\underline{q}\right)}{\partial q_n}-\frac{\partial G\left(\underline{p},\underline{q}\right)}{\partial p_n}\frac{\partial F\left(\underline{p},\underline{q}\right)}{\partial q_n}\right].$$

And let us recall that the evolution of any function $F\left(\underline{p},\underline{q}\right)$ of the canonical coordinates is determined by the equation

$$F'=\left[H,F\right],$$

where the appended prime denotes differentiation with respect to the "timelike" variable corresponding to the evolution induced by the Hamiltonian *H*.

It is now convenient to introduce the "relative coordinates" x_n and the "relative momenta" y_n via the standard definitions

$$x_n = q_n - Q, \; y_n = p_n - \frac{P}{N} .$$

Note that these are not canonically conjugated quantities, since $[y_n, x_m] = \delta_{nm} - 1/N$, and they are not independent since obviously their sum vanishes:

$$\sum_{n=1}^{N} y_n = 0, \; \sum_{n=1}^{N} x_n = 0.$$

It is moreover convenient to introduce the “relative-motion” Hamiltonian $h(\underline{y}, \underline{x})$ via the formula

$$h(\underline{y}, \underline{x}) = \frac{1}{2} \sum_{n=1}^{N} y_n^2 + V(\underline{x}) = \frac{1}{4N} \sum_{n,m=1}^{N} (p_n - p_m)^2 + V(\underline{q})$$

so that

$$H(\underline{p}, \underline{q}) = \frac{P^2}{2N} + h(\underline{y}, \underline{x}).$$

Note that this definition of the relative-motion Hamiltonian $h(\underline{y}, \underline{x})$ entails that it Poisson commutes with both *P* and *Q* :

$$[P, h] = 0, \; [Q, h] = 0.$$

For completeness and future reference let us also display the equations of motion implied by the original Hamiltonian $H(\underline{p},\underline{q})$:

$$q_n' = p_n\,,\ \ p_n' = -\partial V(\underline{q})/\partial q_n\,,\ \ q_n'' = -\partial V(\underline{q})/\partial q_n\,,$$

where (for reasons that will be clear below) we denote as τ the independent variable corresponding to this Hamiltonian flow and with appended primes the differentiations with respect to this variable:

$$q_n \equiv q_n(\tau),\ \ p_n \equiv p_n(\tau),\ \ q_n' \equiv \partial q_n(\tau)/\partial\tau\,,\ \ p_n' \equiv \partial p_n(\tau)/\partial\tau\,.$$

Hence

$$Q' = \frac{P}{N}\,,\ \ P' = 0$$

yielding

$$Q(\tau) = Q(0) + \frac{P(0)}{N}\tau\,,\ \ P(\tau) = P(0),$$

as well as

$$x_n' = y_n = \frac{\partial h(\underline{y},\underline{x})}{\partial y_n}\,,\ \ y_n' = -\frac{\partial V(\underline{x})}{\partial x_n} = -\frac{\partial h(\underline{y},\underline{x})}{\partial x_n}\,.$$

Note that these equations have the standard Hamiltonian form even though, as mentioned above, x_n and y_n are not canonically conjugated variables.

This ends the review of quite standard results for the classical nonrelativistic many-body problem. Let us also emphasize that, above and below, the restriction to unit-mass particles, and to one-dimensional space, is merely for simplicity: generalizations – also of the following results – to the more general case with *different* masses and *arbitrary* space dimensions is quite elementary, essentially trivial.

The isochronous Hamiltonian

The Ω-modified *isochronous* Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ is now defined by the formula

$$\tilde{H}(\underline{p},\underline{q};\Omega)=\frac{1}{2}\left\{\left[P+\frac{h(\underline{y},\underline{x})}{b}\right]^2+\Omega^2 Q^2\right\},$$

where b is an arbitrary constant (introduced for dimensional reasons: it has the dimensions of a momentum, hence of the square-root of an energy) and Ω is a *positive* constant. Let us emphasize that hereafter the evolution of the various quantities is that caused by this new Ω-modified Hamiltonian; the corresponding independent variable is hereafter denoted as t (and interpreted as "time"), and differentiations with respect to this variable will be denoted, as usual, by superimposed dots, and of course, for any function $F\equiv F(\underline{p},\underline{q})$ of the canonical variable its time evolution will be determined by the standard equation

$$\dot{F}=\left[\tilde{H},F\right].$$

Solution of the isochronous Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$

$$Q(t)=Q(0)\cos(\Omega\,t)+\dot{Q}(0)\frac{\sin(\Omega\,t)}{\Omega}=bC\,\frac{\sin[\Omega(t-t_0)]}{\Omega}\,,$$

$$P(t)=P(0)\cos(\Omega t)+\dot{P}(0)\frac{\sin(\Omega t)}{\Omega}+\frac{h[\underline{y}(0),\underline{x}(0),]}{b}[\cos(\Omega t)-1],$$

$$\tilde{x}_n(t)=x_n(\tau),\;\;\tilde{y}_n(t)=y_n(\tau),$$

where (changing for convenience notation) we now denote as $\tilde{x}_n\,,\tilde{y}_n$ the canonical variables whose time evolution is determined by the Ω-modified Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ and as $x_n\,,y_n$ the canonical variables whose time evolution is determined by the original, un-modified Hamiltonian $H\left(\underline{p},\underline{q}\right)$. And here (most importantly)

$$\tau\equiv\tau(t)=\{A\sin(\Omega\,t)+B[1-\cos(\Omega\,t)]\}/\,\Omega\;,$$

where the constants *A* and *B* are given by simple explicit formulas in terms of the initial position and velocity of the centre-of-mass of the system and of the Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ (which is of course a constant of motion). The crucial observation is that $\tau(t)$ (hence the entire solution) is a periodic function of *t* with period

$$T=2\pi\,/\,\Omega\;.$$

Behavior of the *isochronous* system over time scales much shorter than *T*

The solution formulas displayed above demonstrate that the dynamics yielded by the Hamiltonian $\tilde{H}(\underline{p},\underline{q};\Omega)$ does not differ --- on a time scale short with respect to the period *T* --- from that yielded by the original Hamiltonian $H(\underline{p},\underline{q})$ (up to a *constant* rescaling of time). Indeed clearly $\tau=\tau(t)$ on a sufficiently short time scale varies linearly in *t,* since in the neighbourhood of any time $\bar{t}$ --- except when $\dot{\tau}(\bar{t})=C\cos(\Omega\bar{t})$ vanishes ---

$$\tau(t)=\overline{C}+t\,C\cos(\Omega\bar{t})+O\left[\left(\frac{t-\bar{t}}{T}\right)\right],$$

$$\overline{C}=C\left[\sin(\Omega\bar{t})-\Omega\bar{t}\cos(\Omega\bar{t})\right]/\Omega\ .$$

Transient chaos

One therefore finds that – essentially throughout the time evolution -- the Ω-modified dynamics differs from the unmodified one solely by a time rescaling -- by a possibly negative coefficient -- and by a time shift. The coefficient and the shift are time-independent over a time scale much smaller than the *isochrony* period $T=2\pi/\Omega$, but vary periodically with period T. A peculiar state of affairs arises, however, whenever $d\tau/dt$ changes its sign: this of course happens twice within every time period T, this being in fact a consequence of the periodicity of $\tau(t)$, which itself is the cause of the *isochrony*.

It is interesting to speculate on the application of this Ω-modification technique to any (translation-invariant) Hamiltonian describing a “realistic” translation-invariant many-body problem featuring, in its centre-of-mass system, “chaotic” motions with a natural time scale T_C. Then --- provided the constant Ω is assigned so that the *isochrony* period $T=2\pi/\Omega$ is very much larger than this time scale, $T>>T_C$ --- the Ω-modified problem shall exhibit some kind of *chaotic* behavior for quite some time before the *isochronous* character of all its motions takes over, causing thereafter a recurrent evolution. This phenomenology --- qualitative rather than quantitative as described here, since a precise definition of *chaos* requires generally that a system displaying it be observed for *infinite* time --- is nevertheless remarkable.

The quantum case

Finally we tersely show that, in a quantal context, our Ω-modified Hamiltonian

$$\tilde{H}\left(\underline{p},\underline{q};\Omega\right)=\left\{\left[P+h\left(\underline{y},\underline{x}\right)/b\right]^2+\Omega^2\,Q^2\right\}/2\,,$$

features an (infinitely degenerate) equispaced spectrum with spacing $\hbar\Omega$.

This spectrum consists of the eigenvalues E_k of the stationary Schrödinger equation

$$\frac{1}{2}\left\{\left[-i\hbar\frac{\partial}{\partial Z}+\frac{\lambda}{b}\right]^2+\Omega^2 Z^2\right\}\Psi_k\left(Z;\lambda\right)\psi_\lambda\left(\underline{z};\lambda\right)=E_k\Psi_k\left(Z;\lambda\right)\psi_\lambda\left(\underline{z};\lambda\right),$$

obtained from this Hamiltonian via the standard quantization rule

$$P\Rightarrow -i\hbar\partial/\partial Z\,,\;\; Q\Rightarrow Z\;;\;\; \underline{p}\Rightarrow -i\hbar\partial/\partial\underline{z}\,,\;\; \underline{q}\Rightarrow\underline{z}\quad ,$$

and by identifying λ as an eigenvalue of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. Indeed this Schrödinger equation is obtained by assuming that the eigenfunctions of the quantized version of the Hamiltonian $\tilde{H}\left(\underline{p},\underline{q};\Omega\right)$ factor into the product of an eigenfunction, $\Psi_k\left(Z;\lambda\right)$, depending on the variable Z and on which acts the differential operator $\partial/\partial Z$, and of the eigenfunction $\psi_\lambda\left(\underline{z};\lambda\right)$,corresponding to the eigenvalue λ of the quantized version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$. The justification for this factorization is in the commutativity of the operators representing the quantal versions of the canonical variables P and Q, see above, with the operator representing the quantal version of the relative-motion Hamiltonian $h\left(\underline{y},\underline{x}\right)$ -- a commutativity reflecting the Poisson-commutativity of the corresponding quantities in the classical context.

It is now plain that the above Schrödinger equation features the spectrum and eigenfunctions

$$E_k=\hbar\Omega\left(k+1/2\right),\;\; k=0,1,2,...,$$

$$\psi_k\left(Z;\lambda\right)=\exp\left(\frac{i\lambda z}{b\sqrt{\hbar\Omega}}-\frac{z^2}{2}\right)H_k\left(z\right),\;\; z=Z\sqrt{\frac{\Omega}{\hbar}}\quad ,$$

where $H_k\left(z\right)$ denotes the standard Hermite polynomial of order k. This spectrum is of course equispaced with spacing $\hbar\Omega$, and it is infinitely degenerate inasmuch as it does not feature any dependence on the eigenvalues λ.

How to extend an *arbitrary* dynamical system so that the dynamics of the extended system is *isochronous* with period T , and moreover, over a fraction $(1-\varepsilon)\,T$ of that period (with ε arbitrary in the open interval $0<\varepsilon<1$), it is *exactly identical* to that of the original model

$$\underline{X}'=\underline{h}(\underline{X});\;\; X_n'=h_n(\underline{X}),\;\; n=1,\ldots,N\,,\;\; (*)$$

$$\dot{\underline{x}}=\dot{\tau}(t)\,\underline{h}(\underline{x});\;\; \dot{x}_n=\dot{\tau}(t)\,h_n(\underline{x}),\;\; n=1,\ldots,N\,,\;\; (**)$$

$$\underline{x}(t)=\underline{X}[\tau(t)];\;\; x_n(t)=X_n[\tau(t)],\;\; n=1,\ldots,N\,,$$

where $\underline{X}(\tau)$ is the general solution of (*). **Hence** $\tau(t+T)=\tau(t)$ **implies** $\underline{x}(t+T)=\underline{x}(t)$.

In conclusion, whenever $\underline{h}(\underline{X})$ is such that the solution of the system (*) of *N* nonlinear ODEs exists globally, *all* solutions of the system (**) are periodic with period *T*: this system is ***isochronous***.

However, the system (**) is *not* autonomous. To eliminate this "defect", we perform the second step of our treatment, replacing this system (**) with the system

$$\dot{\underline{x}} = \phi \, \underline{h}(\underline{x}); \;\; \dot{x}_n = \phi \, h_n(\underline{x}), \;\; n = 1, \ldots, N, \;\; (***)$$

which is of course equivalent provided the time-evolution of the scalar quantity ϕ is such that

$$\phi(t) = \dot{\tau}(t), \;\; \tau(t) = \int_0^t dt' \phi(t') .$$

To construct a function $\phi(t) = \dot{\tau}(t)$ having the desired properties one may proceed as follows.

$$\dot{f}_1(t) = \Omega f_2(t), \;\; \dot{f}_2(t) = -\Omega f_1(t); \;\; T = 2\pi/\Omega;$$

$$\phi(f_1, f_2; \varepsilon) = 1 - K(\varepsilon) \, \theta[S(f_1, f_2; \varepsilon)] \, \exp\left\{-[S(f_1, f_2; \varepsilon)]^{-2}\right\},$$

$$\theta(x) = 1 \;\; if \;\; x > 0, \;\; \theta(x) = 0 \;\; if \;\; x < 0,$$

$$K(\varepsilon) = \frac{1}{\varepsilon}\left(\int_0^1 dx \, \exp\left\{-\left[\sin^2\left(\frac{\varepsilon\pi}{2}\right) - \sin^2\left(\frac{\varepsilon\pi}{2} x\right)\right]^{-2}\right\}\right)^{-1}, \;\; 0 < \varepsilon < 1,$$

$$S(f_1, f_2; \varepsilon) = -\frac{f_1^{\,2}}{f_1^{\,2} + f_2^{\,2}} + \sin^2\left(\frac{\varepsilon\pi}{2}\right) .$$

Solution:

$$\underline{x}(t) = \underline{X}[\tau(t)];\ \ x_n(t) = X_n[\tau(t)],\ \ n = 1,\ldots,N\ ,$$

$$f_1(t) = A\ \sin(\Omega t - \eta\pi)\ ,\ \ f_2(t) = A\ \cos(\Omega t - \eta\pi)\ ,$$

$$0 \le \eta < 1\ ;\ \ \ f_{1,2}(t \pm T) = f_{1,2}(t)\ ;\ \ \ T = 2\pi / \Omega\ ;$$

$$S[f_1(t), f_2(t); \varepsilon] = S(t) = -\sin^2(\Omega t - \eta\pi) + \sin^2(\varepsilon\pi / 2)\ ,$$

$$S(t \pm T/2) = S(t),\ S(T_\pm) = 0,\ T_\pm = (\eta \pm \varepsilon/2) T/2\ ;$$

$$\phi(t) = 1 - \theta[S(t)]\ \dot{\Psi}(t)\ ,\ \ \Psi(t) = K(\varepsilon)\int_0^t dt' \exp\left\{-[S(t')]^{-2}\right\}\ ,$$

$$\phi(t \pm T/2) = \phi(t),\ \ \int_0^{T/2} dt\ \phi(t) = 0\ ;\ \ \tau(t \pm T/2) = \tau(t)\ .$$

If $\eta \geq \varepsilon/2$, hence $0 \leq T_- < T_+ < T/2$, then

$$\tau(t) = t \quad for \qquad 0 \leq t < T_- \; ,$$

$$\tau(t) = t - \Psi(t) + \Psi(T_-) \quad for \quad T_- \leq t < T_+ \; ,$$

$$\tau(t) = t - T/2 \quad for \qquad T_+ \leq t < T/2 \; ;$$

if instead $\eta \leq \varepsilon/2$, hence $T_- \leq 0 < T_+ < T/2$, then

$$\tau(t) = t - \Psi(t) + A \quad for \quad 0 \leq t < T_+ \; ,$$

$$\tau(t) = t + A - B \quad for \quad T_+ \leq t < T_- + T/2 \; ,$$

$$\tau(t) = t - \Psi(t) + A - B + C \quad for \quad T_{-.} + T/2 \leq t < T/2 \; ;$$

$$A = \Psi(0), \; B = \Psi(T_+)_-, \; C = \Psi(T_- + T/2) \, .$$

Note that in both cases $\tau(t)$, besides being *periodic* with period *T/2*, is *linear* in *t* over a fraction $1 - \varepsilon$ of its period *T/2*.

Outlook

We have seen that, given an arbitrary (autonomous) dynamical system --- possibly characterized by a *chaotic* dynamics (according to the standard definitions of *chaotic* behavior) --- it is possible to extend it so as to obtain thereby a new (autonomous) dynamical system which is *isochronous* (all its solutions are completely periodic with an *a priori* fixed period) hence it is *integrable* (indeed, “more than superintegrable”: according to the standard definition of superintegrable systems, as possessing the maximal number of functionally independent constants of motion, so that *all* their confined solutions are *completely periodic*, but not necessarily *isochronous*); yet it is also such that, over an *arbitrarily large* fraction (of course, less than unity) of its period it reproduces *exactly* the dynamics of the original model. This fact suggests the need to invent new definitions of *chaotic* behavior (or perhaps, using a new name, of “*complex*” behavior) which do not refer to the behavior of a system over *infinite* time (as the current definition of *chaotic* behavior does), but rather equip this new definition with an associate *time scale* characterizing its validity. Note that, to some extent, this is now being done in the *integrable* case via the introduction of multiple scale analysis and the recognition that a *nonintegrable* system may nevertheless be *more or less nonintegrable* depending on the order of a multiple scale reduction of it that yields an *integrable* dynamics (indeed *all* dynamical systems are *integrable* over a sufficiently short time scale).

Happy Birthdate Volodja !!!