Solitons, boundary value problems and a nonlinear method of images

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in honor of Prof. Zakharov's 70th birthday

Frontiers in Nonlinear Waves, Tucson, AZ - March 26-29, 2010

Introduction: solitons in BVPs?

- In 1972&1973, Zakharov and Shabat solved the IVP for focusing and defocusing NLS via IST. Each soliton corresponds to a discrete eigenvalue of the scattering problem.

- But BVPs still offer some surprises.

- Main approaches to the BVP:
  - Extension to the whole line [Ablowitz-Segur, Fokas, Tarasov, Habibullin]
  - Simultaneous spectral analysis of Lax pair + global relation [Fokas]
  - Other methods [Degasperis-Manakov-Santini, Sabatier. . .]

- In all these approaches:
  - the problem only truly becomes linear for certain kinds of BCs, called linearizable BCs.
  - the relation between solitons and discrete eigenvalues is exactly the same as in the IVP.

- But there are some issues. . .

NLS solitons on the half line?

- NLS on the half line:

$$iq_t + q_{xx} - 2\nu|q|^2 q = 0\,, \qquad 0<x<\infty, \quad t>0\,,$$

$q(x,0)$ given $\quad (\nu = \mp 1,$ focusing/defocusing)

- Linearizable BCs: homogeneous Robin,

$$q_x(0,t) - \alpha\, q(0,t) = 0\,, \qquad \alpha \in \mathbb{R}\,.$$

Special cases:

$\alpha \to \infty$: Dirichlet BCs, $q(0,t) = 0$.
$\alpha = 0$: Neumann BCs, $q_x(0,t) = 0$.

- In the IVP, a discrete eigenvalue $k_j = (V_j + iA_j)/2$ yields a soliton

$$q_s(x,t;A_j,V_j,\xi_j,\varphi_j) = 2A_j\,\mathrm{sech}[2A_j(x - V_j t - \xi_j)]\,\mathrm{e}^{i[V_j x + 2(A_j^2 - V_j^2)t + \varphi_j]}\,.$$

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- But the soliton solution does not satisfy the BCs.
  So, the BVP admits no soliton solutions?

- Numerical solutions show that solitons are reflected at the boundary.
  But, the soliton velocity is the real part of the conserved eigenvalue!

Outline

0. Introduction

1. IST for NLS on the whole line

2. NLS on the half line with linearizable BCs
   - Dirichlet BCs
   - Neumann BCs
   - Robin BCs
   - Relations between eigenvalues and norming constants

3. Soliton behavior
   - Soliton reflection
   - Multi-soliton solutions
   - Reflection-induced shift

4. Soliton reflection in other settings
   - Defocusing NLS with non-zero BCs at infinity
   - Ablowitz-Ladik lattice on the natural numbers

5. Conclusions

Outline

IVP for NLS: eigenfunctions

- Matrix Lax pair for NLS:

$$\mu_x - ik[\sigma_3, \mu] = \mathrm{Q}\mu\,, \quad \mu_t + 2ik^2[\sigma_3, \mu] = -\big(i\mathrm{Q}\mathrm{Q}\,\sigma_3 + i\mathrm{Q}_x\sigma_3 + 2k\mathrm{Q}\big)\mu\,,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{Q}(x,t) = \begin{pmatrix} 0 & q \\ \nu q^* & 0 \end{pmatrix}.$$

- Jost solutions: $\mu^{(1,2)}(x,t,k)$ s.t.

$$\lim_{x\to-\infty} \mu^{(1)}(x,t,k) = \mathrm{I}\,, \qquad \lim_{x\to\infty} \mu^{(2)}(x,t,k) = \mathrm{I}\,.$$

- Analyticity:

$$\mu^{(1,L)}\;\&\;\mu^{(2,R)}\!:\ \operatorname{Im} k < 0\,, \qquad \mu^{(1,R)}\;\&\;\mu^{(2,L)}\!:\ \operatorname{Im} k > 0\,.$$

- Scattering matrix: (time independent)

$$\mu^{(1)}(x,t,k) = \mu^{(2)}(x,t,k)\,\mathrm{e}^{i\theta\sigma_3}\mathrm{A}(k)\mathrm{e}^{-i\theta\sigma_3} \quad \forall k \in \mathbb{R}\,,$$

$\theta(x,t,k) = kx - 2k^2t$.

- Analyticity: $a_{11}(k)$: LHP, $a_{22}(k)$: UHP, $a_{12}(k), a_{21}(k)$: none.
- Symmetry $k \to k^*$ of the eigenfunctions. As a result,

$$a_{22}(k) = a_{11}^*(k^*) =: a(k), \qquad a_{12}(k) = \nu a_{21}^*(k) =: b(k)\,.$$

Discrete eigenvalues, inverse problem and solution

- Discrete eigenvalues: (bounded eigenfunctions)
  $k_1,\dots,k_J$ in UHP s.t. $a_{22}(k_j)=0$: $\mu^{(1,R)}(x,t,k_j)=b_j e^{2i\theta(k_j)}\mu^{(2,L)}(x,t,k_j)$,
  $\bar k_1,\dots,\bar k_J$ in LHP s.t. $a_{11}(\bar k_j)=0$: $\mu^{(1,L)}(x,t,\bar k_j)=\bar b_j e^{-2i\theta(\bar k_j)}\mu^{(2,R)}(x,t,\bar k_j)$.
  Symmetries: $\bar k_j=k_j^*$ (eigenvalue pairs), $\bar b_j=-b_j^*$.
- Norming constants: (from residue relations)
  $$C_j=b_j/\dot a_{22}(k_j),\qquad \bar C_j=\bar b_j/\dot a_{11}(\bar k_j)=-C_j^*.$$
- Riemann-Hilbert problem:
  $$\mathrm{M}^-=\mathrm{M}^+(\mathrm{I}-\mathrm{R})\quad \forall k\in\mathbb{R},$$
  Sectionally meromorphic matrices:
  $$\mathrm{M}^+(x,t,k)=\big(\mu^{(2,L)}\,,\mu^{(1,R)}/a_{22}\big),\qquad \mathrm{M}^-(x,t,k)=\big(\mu^{(1,L)}/a_{11}\,,\,\mu^{(2,R)}\big).$$
  The jump matrix R contains the reflection coefficient $\rho(k)=b(k)/a(k)$.
- Reconstruction formula: $q(x,t)=-2i\lim_{k\to\infty}k\,M_{12}(x,t,k)$.
  (sum from discrete spectrum plus integral from continuum spectrum).
- Pure soliton solutions $=$ reflectionless solutions $[\rho(k)\equiv 0]$
  $q(x,t)$ is obtained from the solution of an algebraic system.

Outline

Dirichlet BCs

- Define the odd extension of the potential: $\forall x \in \mathbb{R}$,
$$Q^{\text{ext}}(x,t) = Q(x,t)\,\Theta(x) - Q(-x,t)\,\Theta(-x)\,,$$
$\Theta(x) =$ Heaviside step function.
Then use the IST for the IVP.
- Symmetry $x \to -x$ of the potential $\Rightarrow$ symmetry of the eigenfunctions:
$$\mu^{(1)}(x,t,k) = \mu^{(2)}(-x,t,-k)\,,$$
- Combine with usual ones $\Rightarrow$ further symmetry of scattering coeffs:
$$a(-k^*) = a^*(k), \quad b(k) = -b(-k)\,.$$

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- Theorem: the discrete spectrum has the following properties:
  (i) the number J of discrete eigenvalues in the UHP/LHP is even;
  (ii) discrete eigenvalues appear in quartets: $\{\pm k_n\,, \pm k_n^*\}_{n=1}^{N}$,
  Here $N = J/2 =$ number of discrete eigenvalues in each quadrant.

- Label the discrete eigenvalue symmetric to k_n as $k_{n'} = -k_n^*$.
Relation between the norming constants associated with k_n & $k_{n'}$:
$$b_n b_{n'}^* = -1\,, \quad C_n C_{n'}^* = 1/\dot{a}^2(k_n)\,.$$

Neumann BCs

- Use the even extension of the potential:
$$\mathrm{Q}^{\mathrm{ext}}(x,t) = \mathrm{Q}(x,t)\,\Theta(x) + \mathrm{Q}(-x,t)\,\Theta(-x)\,.$$
- Symmetry of eigenfunctions and scattering matrix:
$$\mu^{(1)}(x,t,k) = \sigma_3\mu^{(2)}(-x,t,-k)\,, \quad a(-k^*) = a^*(k),\ b(k) = b(-k)\,.$$
(The symmetries of $b(k)$ are those of the FT of odd/even functions.)
- The same symmetry applies for the discrete spectrum.
(But here self-symmetric eigenvalues can arise: $k_{n'} = k_n$ when $k_n = i\kappa_n$, so J is not necessarily even.)

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(But here self-symmetric eigenvalues can arise: $k_{n'} = k_n$ when $k_n = i\kappa_n$, so J is not necessarily even.)
- Norming constants of symmetric eigenvalues:
$$b_n b_{n'}^* = 1\,,\quad C_n C_{n'}^* = -1/\dot{a}^2(k_n)\,.$$
- One-soliton solutions of the BVP: $N = 1$ & $b(k) = 0$.
Let $k_1 = (V + iA)/2$ and $C_j = A_j e^{A_j\xi_j + i(\varphi_j+\pi/2)}$, $j = 1,2$. Then
$$\xi_1 + \xi_2 = \frac{1}{A}\log\left(1 + \frac{A^2}{V^2}\right),\quad \varphi_2 - \varphi_1 = 2\arg\left(A + i\,V\right) + \nu_\alpha\pi\,,$$
Dirichlet BCs: $\nu_\alpha = 0$; Neumann BCs: $\nu_\alpha = 1$.

Robin BCs

- An extension of the potential is also possible here. Approaches:
  - use a k-dependent extension
  - use a particular kind of auto-Bäcklund transformations
- Either way, one obtains the same result:

$$a(k) = a^*(-k^*), \; b(k) = f(k)b(-k),$$

where

$$f(k) = (2k - i\alpha)/(2k + i\alpha).$$

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- Same symmetries as before apply for the discrete eigenvalues.

$$b_n b_{n'}^* = f(k_n)\,, \quad C_n C_{n'}^* = -f(k_n)/\dot{a}^2(k_n)\,,$$

- No self-symmetric eigenvalues are possible for any $\alpha \neq 0$.
- Pure one-soliton solutions:

$$\xi_1 + \xi_2 = \frac{1}{A}\log\left(1 + \frac{A^2}{V^2}\right) + \frac{1}{2A}\log\left[\frac{V^2 + (A-\alpha)^2}{V^2 + (A+\alpha)^2}\right],$$

$$\varphi_2 - \varphi_1 = 2\arg\left(A + i\,V\right) + \pi - \arg\left[\frac{V + i(A-\alpha)}{V + i(A+\alpha)}\right].$$

Relations between eigenvalues and norming constants

- Recall $C_n C_{n'}^* = -f(k_n)/\dot{a}^2(k_n)$, and let $C_j = A_j e^{A_j \xi_j + i(\varphi_j + \pi/2)}$ as before.
- Recall that the analytic scattering coeffs obey trace formulae:

$$\log a(k) = \sum_{j=1}^{J} \log\left(\frac{k-k_j}{k-k_j^*}\right) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log|a(k')|^2}{k'-k}\,dk'\,.$$

- In particular, for reflectionless solutions,

$$\dot{a}(k_j) = {\prod_{m=1}^{J}}'(k_j - k_m) \Big/ \prod_{m=1}^{J}(k_j - k_m^*)\,,$$

  where the prime denotes that the term with $m = j$ is missing.
- The symmetry of the discrete spectrum then yields

$$\xi_n + \xi_{n'} = (*) - \frac{1}{A_n}{\sum_{m=1}^{N}}' \log \frac{[(V_n - V_m)^2 + (A_n - A_m)^2][(V_n + V_m)^2 + (A_n - A_m)^2]}{[(V_n + V_m)^2 + (A_n + A_m)^2][(V_n - V_m)^2 + (A_n + A_m)^2]}\,,$$

$$\varphi_n - \varphi_{n'} = (*) - 2{\sum_{m=1}^{N}}' \arg \frac{[V_n - V_m + i(A_n - A_m)][V_n + V_m + i(A_n - A_m)]}{[V_n + V_m + i(A_n + A_m)][V_n - V_m + i(A_n + A_m)]}\,,$$

  where $k_n = (V_n + iA_n)/2$,
  $(*)$ denotes the terms in the one-soliton case.

Outline

Soliton reflection

- physical solitons = solitons located at $x > 0$,
  mirror solitons = their symmetric counterparts (since they can be thought of as reflected images).
- Since $k_j = (V_j + iA_j)/2$ and $k_{n'} = -k_n^*$, it is $A_{n'} = A_n$ and $V_{n'} = -V_n$.
- The apparent soliton reflection at the boundary is a soliton interaction in which norming constants are s.t. the appropriate BCs are satisfied.
- After the collision the solitons re-emerge intact (as usual), but the role of physical and mirror soliton has been swapped.

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- After the collision the solitons re-emerge intact (as usual),
  but the role of physical and mirror soliton has been swapped.
- WLOG, label the discrete eigenvalues s.t. $V_1 \leq \cdots \leq V_{2N}$.
  ($V_j < 0 \Rightarrow$ left-going solitons; $V_j > 0 \Rightarrow$ right-going solitons.)
- As $t \to -\infty$, the physical solitons have eigenvalues $k_1, \ldots, k_N$,
  the mirror solitons $k_{N+1}, \ldots, k_{2N}$.

  As each soliton is reflected, k_n and $k_{n'} = k_{2N-n+1}$ switch role.

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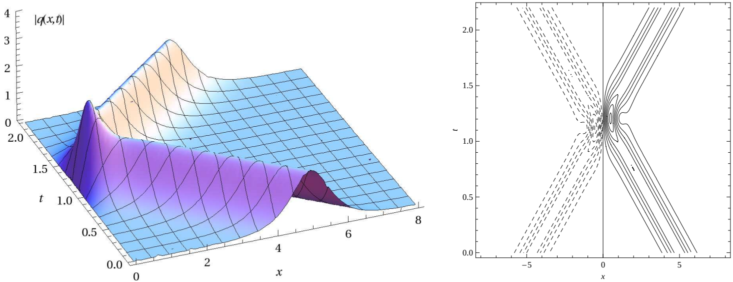
  the mirror solitons $k_1, \ldots, k_N$.

Soliton reflection: Dirichlet BCs

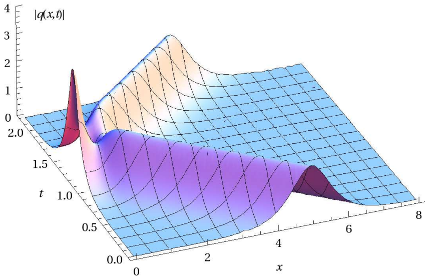


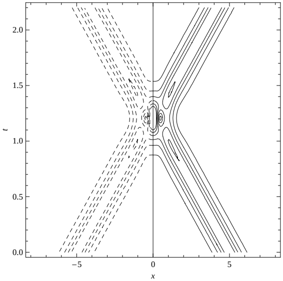
$A = 2$, $V = -2$, $\xi = 5$, $\varphi = 0$.

Left: 3D plot of $|q(x,t)|$.
Right: contour plot including the mirror soliton to the left of the boundary.
Note: these and all other figures show exact solutions, not numerical simulations.

Note that a mirror soliton is needed to satisfy the BCs even when $V > 0$.
Of course in this case the soliton reflection occurs at $t < 0$.

Soliton reflection: Neumann BCs





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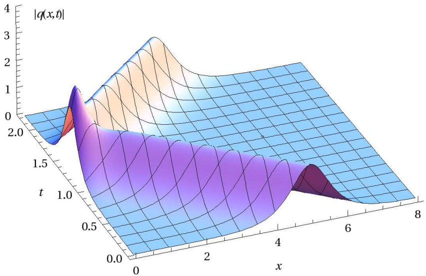
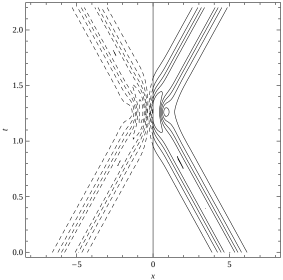
Soliton reflection: Robin BCs



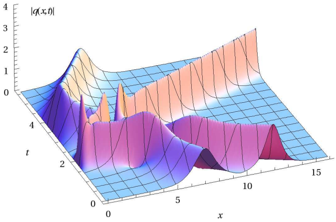


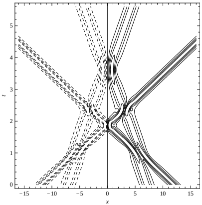
$\alpha = 3$; $A = 2$, $V = -2$, $\xi = 5$, $\varphi = 0$.

(Here for $x < 0$ the contour plot shows the analytic continuation of the solution to the negative real axis using Bäcklund transformations, not $q^{\text{ext}}(x, t, k)$.)

Soliton reflection: $N > 1$

These results are not limited to one-soliton solutions of the BVP, nor to reflectionless solutions:





Top: $A_1 = 2$, $A_2 = 3/2$, $V_1 = -3$, $V_2 = -1$, $\xi_1 = 12$, $\xi_2 = 8$, $\varphi_1 = \varphi_2 = 0$, Dirichlet BCs.

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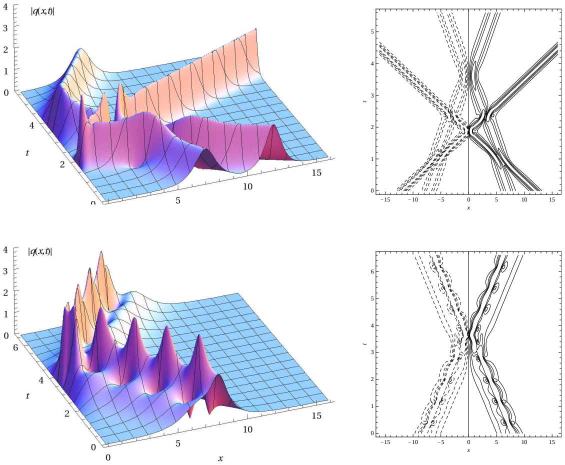

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Bottom: $A_1 = 1$, $A_2 = 3$, $V_1 = V_2 = -1$, $\xi_1 = \xi_2 = 10$, $\varphi_1 = \varphi_2 = 0$, Dirichlet BCs.

Reflection-induced shift

- Since any soliton interaction yields a position shift, so does the reflection.
- Interaction-induced position shift: $\delta\xi_j = \xi_j^+ - \xi_j^-$.
  Compare long-time asymptotics $q(x,t) \underset{t\to\pm\infty}{\sim} \sum_{j=1}^{J} q_s[x,t;A_j,V_j,\xi_j^\pm,\varphi_j^\pm]$.
- WLOG, take $V_n < 0$: the position shift resulting from the interaction of a physical soliton with its mirror is

$$\delta\xi_{n,n'} = -(1/A_n)\log\left(1 + A_n^2/V_n^2\right) = -\delta\xi_{n',n}\,.$$

- But since the physical and mirror solitons interchange roles, a second contribution exists to the total reflection-induced position shift:

$$\Delta\xi_n = \xi_n^- + \xi_{n'}^- - \delta\xi_n\,,$$

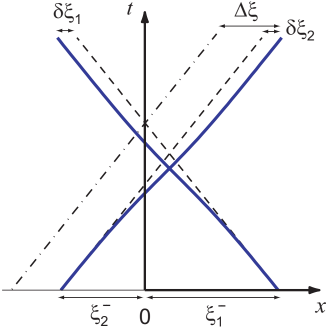
- Simplest case: $N=1$:

$$\Delta\xi = \frac{1}{A}\left[\log\left(\frac{1}{A^2}+\frac{1}{V^2}\right) + \frac{1}{2}\log\left(\frac{V^2+(A-\alpha)^2}{V^2+(A+\alpha)^2}\right)\right].$$

  An explicit formula also exists $\forall N \in \mathbb{N}$.
- Importantly, the reflection shift depends on the BCs! (Cf. $N=1$ figs.)

Reflection-induced shift (cont')

Left: the contributions to the total reflection-induced shift.



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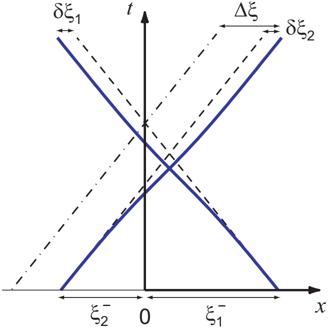

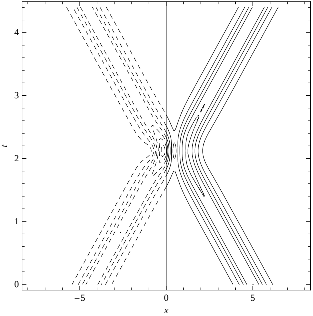

Right: a soliton reflection with a large, BC-induced shift.
($A = -\alpha = 2$, $V = -1$, $\xi = 5$ and $\varphi = 0$)

Outline

Defocusing NLS with non-zero BCs at infinity

Recall that in the IVP for the defocusing case with $|q(x,t)| \to q_o$ as $|x| \to \infty$:
- continuous spectrum: $(-\infty, -q_o] \cup [q_o, \infty)$;
- discrete spectrum $\{k_1, \ldots, k_J\} \subset (-q_o, q_o)$;
- $k_j = 0$: black soliton; $k_j \neq 0$: gray soliton.

In the BVP:
- discrete eigenvalues appear in symmetric pairs: $k_{n'} = -k_n$; corresponding symmetry relations exist for the norming constants.
- a black soliton (self-symmetric eigenvalue) always exists with Dirichlet BCs.
- no black soliton exists for Neumann BCs or Robin BCs.

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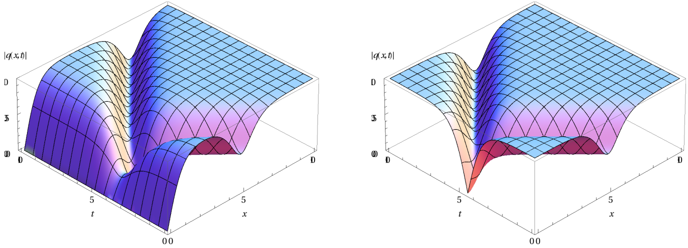

Left: 1 gray soliton (plus the always present black soliton). Right: Neumann BCs; 1 gray soliton.

AL on the natural numbers with certain linearizable BCs

Ablowitz-Ladik lattice:

$$i\dot{q}_n + (q_{n+1} - 2q_n + q_{n-1})/h^2 + |q_n|^2(q_{n+1} + q_{n-1}) = 0.$$

Scattering problem:

$$\Phi_{n+1} = \left[\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} + \begin{pmatrix} 0 & q_n \\ r_n & 0 \end{pmatrix} \right] \Phi_n \,.$$

IVP: discrete eigenvalues appear in quartets:

$$\{z_j, z_j^*, 1/z_j, 1/z_j^*\}, \text{ with } |z_j| \le 1.$$

BVP: Discrete eigenvalues appear in octets: $z_{n'} = -z_n^*$,
and corresponding relations exist for the norming constants.

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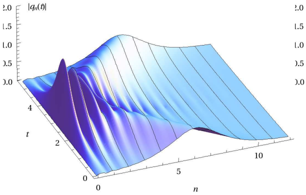
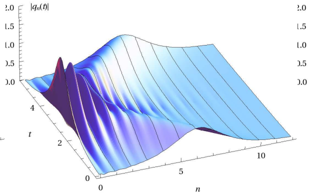
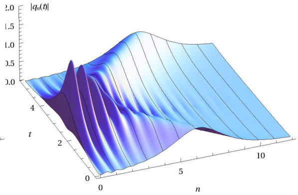


Left: BC $q_0(t) = 0$.



Center: BC $q_1(t) = q_{-1}(t)$.



Right: BC $q_0(t) - q_1(t) = 0$.

Outline

Conclusions

- Main results:
  - discrete eigenvalues appear in symmetric pairs/quartets;
  - norming constants of symmetric eigenvalues are uniquely determined;
  - soliton reflection is the role swap of physical and mirror soliton;
  - a reflection-induced shift exists, with two components;
  - results apply with any number of solitons and with radiation (but only for linearizable BCs).

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  But, unlike electrostatics, here the symmetries of discrete eigenvalues and norming constants are not assumed a priori, but rather derived as a necessary consequence of the BCs via the IST.
- Same phenomena arise in focusing/defocusing NLS and AL.
  The method applies to some other integrable NLEEs, (e.g., SG, Toda...), but probably not to those w/o reflection symmetry (as w/ linear PDEs).
  Also, what happens to solitons for BVPs on finite domains?

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(plus many others omitted due to space constraints)