Nonlinear Waves: from Beaches to Lasers

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Outline

- Introduction
  - Historical remarks
  - Localized nonlinear waves: solitons
- Applications of Nonlinear waves
  - Water waves
  - Nonlinear optics: communications, lasers, photonic lattices
- Conclusion

Introduction

- Linear waves: small amplitude; nonlinear waves: amplitude not small
- A class of waves that have been of long standing interest: *SOLITARY* waves or *SOLITONS*
  They are localized, stable, nonlinear waves
- In math: solitons arise in nonlinear equations which are often hard to analyze
- In physical problems localized waves are frequently observed

Historical Timeline

Solitary waves/solitons first discovered in water waves–brief historical timeline follows

1757 –Fundamental equations of (inviscid) fluids –Leonard Euler (Swiss)–Euler blind for last 20 years of his life wrote almost 900 papers/books -many when he had little or no vision!

Known for many contributions in Mathematics and Physics

Historical Timeline–con't

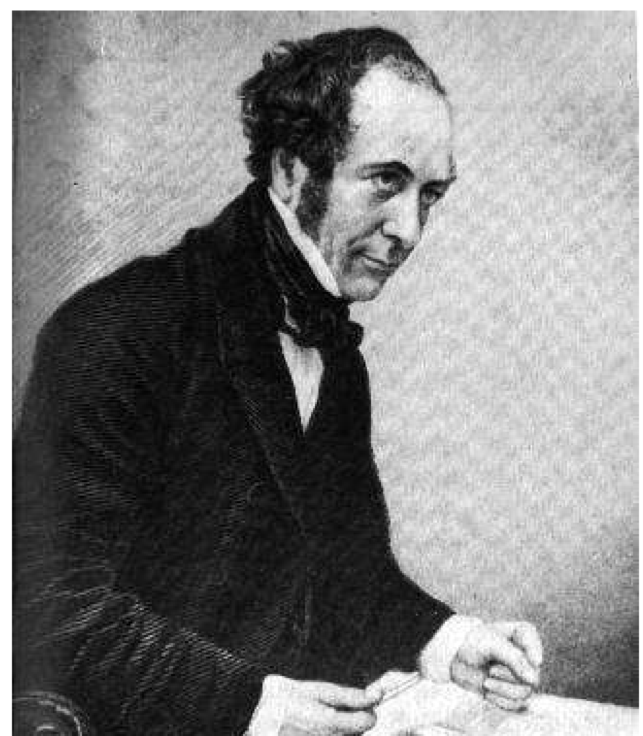
- 1813– French Academy of Sciences announced prize competition-propagation of water waves
- 1816- Augustin Cauchy (French) awarded the prize-linear eq. (initial value problem); Simeon Poisson–a judge of the committee also submitted an important paper
- Cauchy's research was eventually published in 1827; closely related to work by Jean Fourier–Fourier transforms
- Complex work, controversial; Poisson's work published earlier

Fourier, Cauchy and Poisson

Historical Timeline-con't

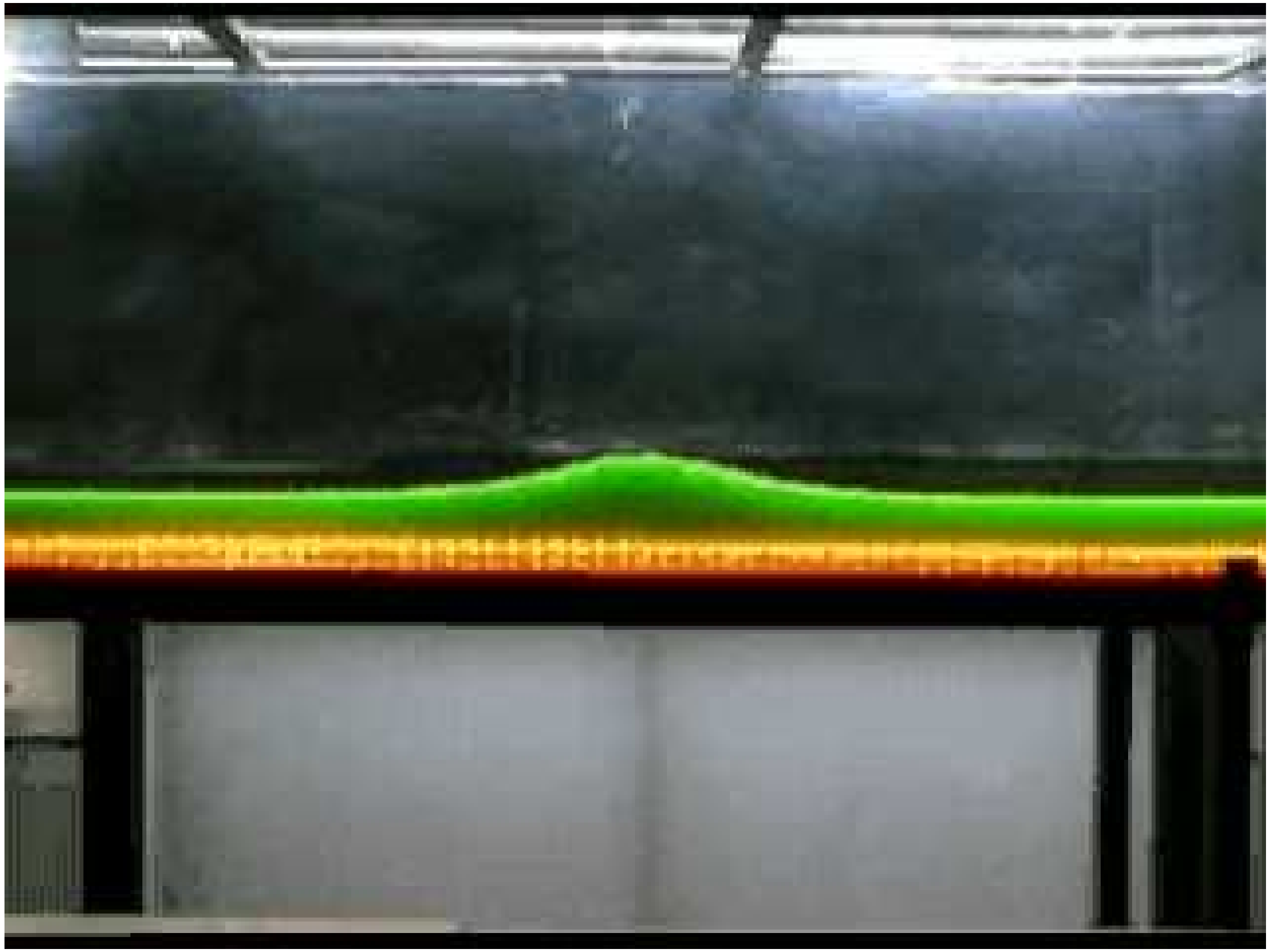
- 1837–British Association for the Advancement of Science (BAAS) sets up a "Committee on Waves"; one of two members was John Scott Russell (1808-1882; Scottish Navel Scientist). Russell (Scottish) was interested in the efficient design of boats.
- 1837, 1840, 1844 (Russell's major effort): "Report on Waves" to the BAAS–describes a remarkable discovery.



Russell-Wave of Translation

- Russell observed a localized (solitary) wave: “rounded smooth...well-defined heap of water”
- Called it the “great primary wave of elevation”
- “ Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon...”

Solitary wave video



Russell: to Mathematicians

"... it now remained for the mathematician to predict the discovery after it had happened..."

Leading British fluid dynamics researchers doubted the importance of Russell's solitary wave. George Airy 1845: wave was linear.

Historical Timeline-con't

1847–George Stokes (Irish): Stokes set up the correct nonlinear water wave equations and found a traveling periodic wave to these equations where the speed depends on amplitude. Stokes made many other critical contributions to fluid dynamics –"Navier-Stokes equations".

Historical Timeline-con't

- 1871-77 – Joseph Boussinesq (French- left): new nonlinear eqs. and solitary wave solution for shallow water waves
- 1895 –Diederick Korteweg (Dutch- right) & Gustav deVries: simplified shallow water wave eq. ("KdV" eq.); nonlinear periodic solution: "cnoidal" wave; special case being the solitary wave solution. Russell's work was (finally) confirmed

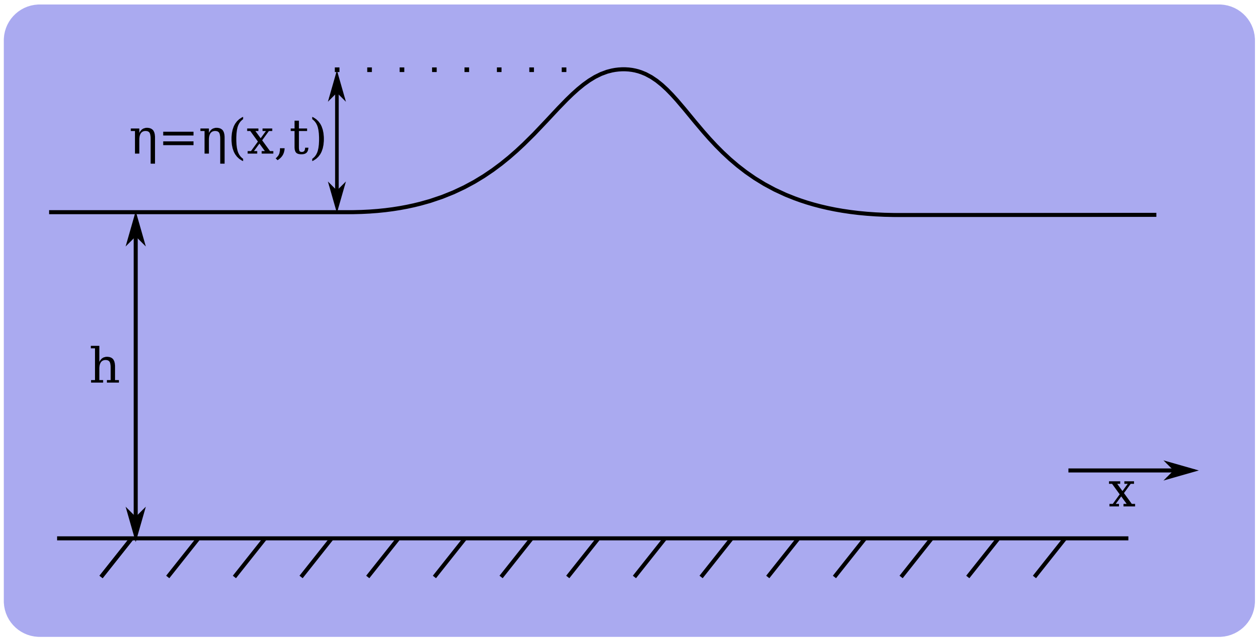
KdV Equation

KdV eq. is a one dimensional nonlinear partial differential equation which approximately described the wave elevation ($\eta = \eta(x,t)$) above mean height (h). The equation also depends on gravity: g and surface tension: T. It has a solitary wave solution. The eq. derived by Korteweg-deVries

$$\frac{1}{\sqrt{gh}}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \frac{h^2}{2}(\frac{1}{3} - \hat{T})\eta_{xxx} = 0$$

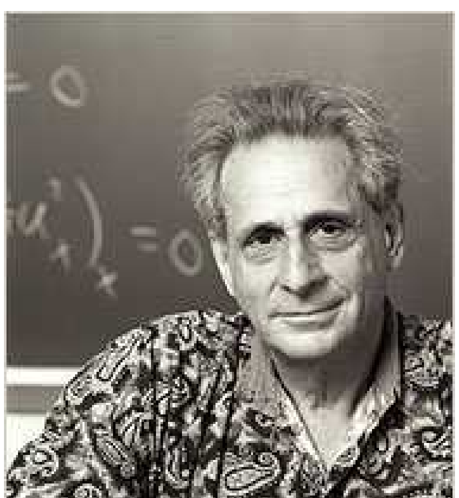
where $\hat{T}$ is normalized surface tension.

Fig. Re: KdV Equation



KdV –Modern Times

- 1895-1960 – Korteweg & deVries (KdV): water waves...
- 1960's– mathematicians developed approximation methods to find reduced equations governing physical systems; KdV and nonlinear Schrödinger (NLS) equations are two important and “universal” ones
- 1965 –computation on KdV eq. (Martin Kruskal and Norman Zabusky) –introduced the term “Solitons”
- 1967 – Method to find solution of KdV (C. Gardner, J. Greene, M. Kuskal - photo below, R. Miura)



KdV – IST

- 1970's-present – KdV developments led to new methods & results in math physics
- Researchers worldwide (US, Russia, Japan, UK, ...) made many important discoveries
- Applications to a number of physically interesting systems
- Termed Inverse Scattering Transform (IST)–find solitons as special solutions
- IST: Ref. Ablowitz & Segur 1981; Ablowitz & Clarkson 1991; Ablowitz, Prinari, & Trubatch 2003: 1+1, 2+1 dimensions, continuous and discrete; many other important texts
- Inverse Scattering Transform: now widely known

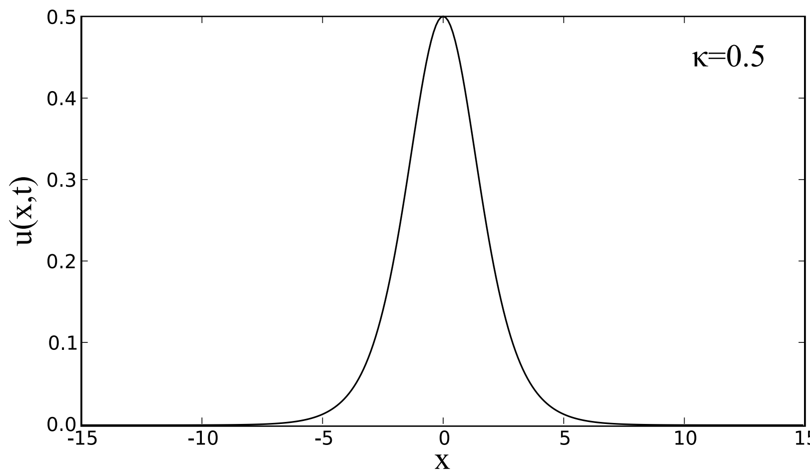
KdV Solitary-Soliton Wave

Normalized equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

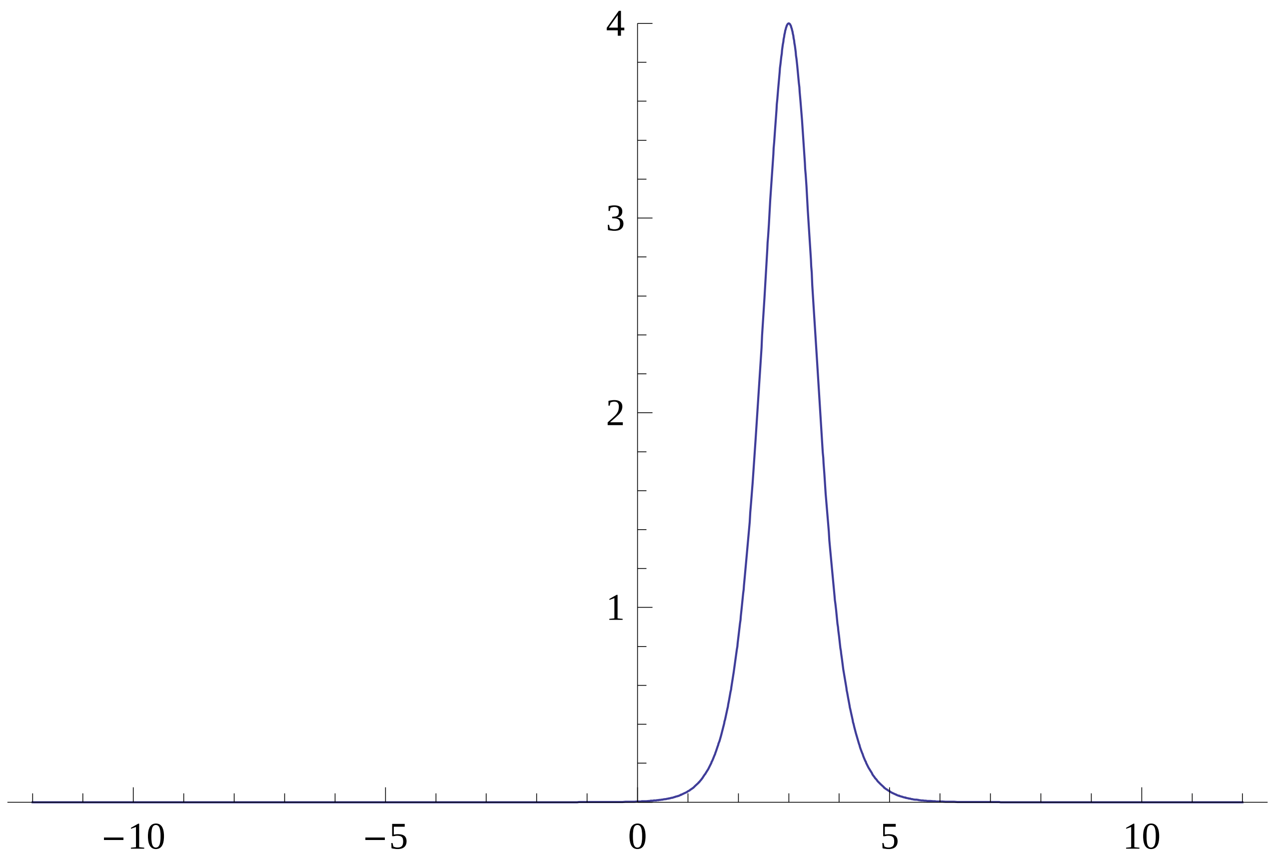
Solitary wave: $u_s(x,t) = 2\kappa^2 \operatorname{sech}^2 \kappa(x - x_0 - 4\kappa^2 t)$

$u_{max} = 2\kappa^2$; speed = $2u_{max}$



Solitons: One Soliton

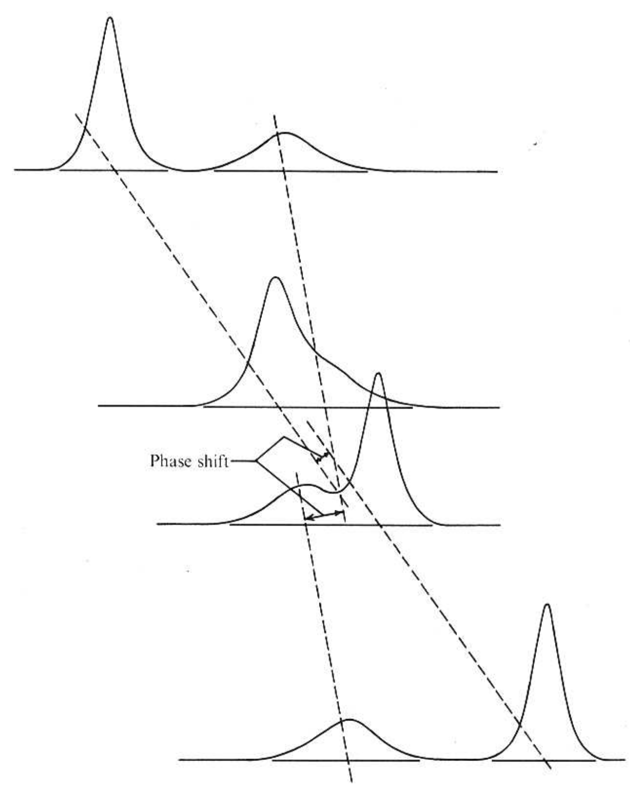
IST: one "eigenvalue"



KdV soliton-physical

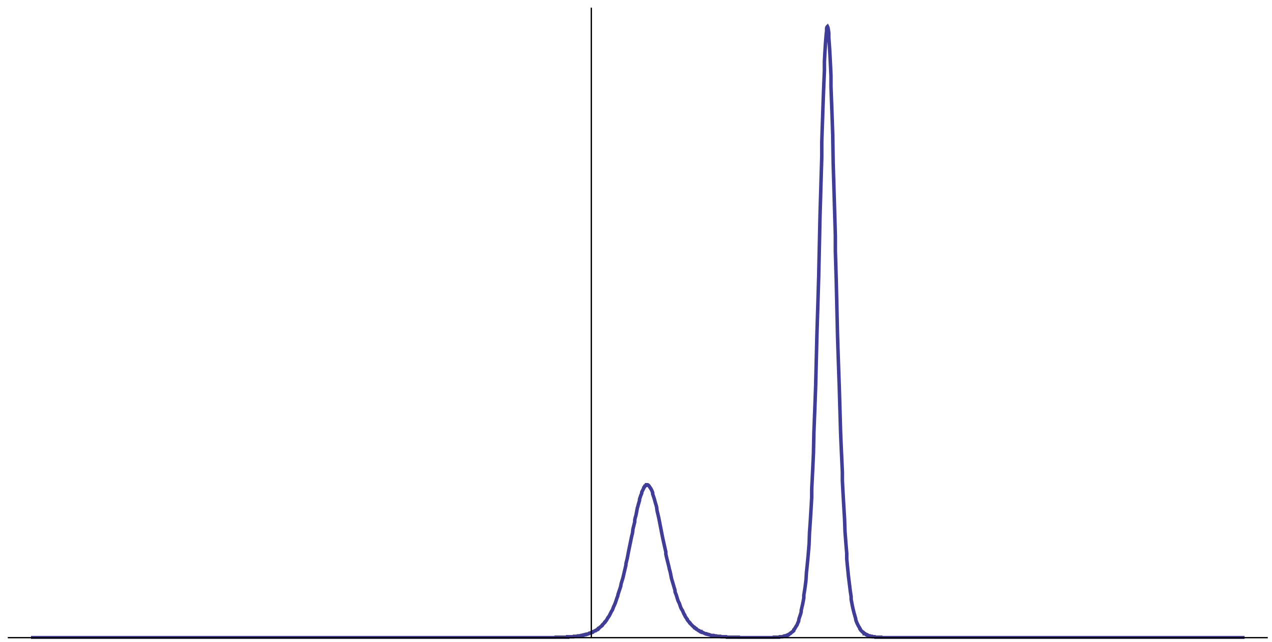
KdV –Soliton Interaction

KdV eq. with two special numbers (two eigenvalues): two solitons

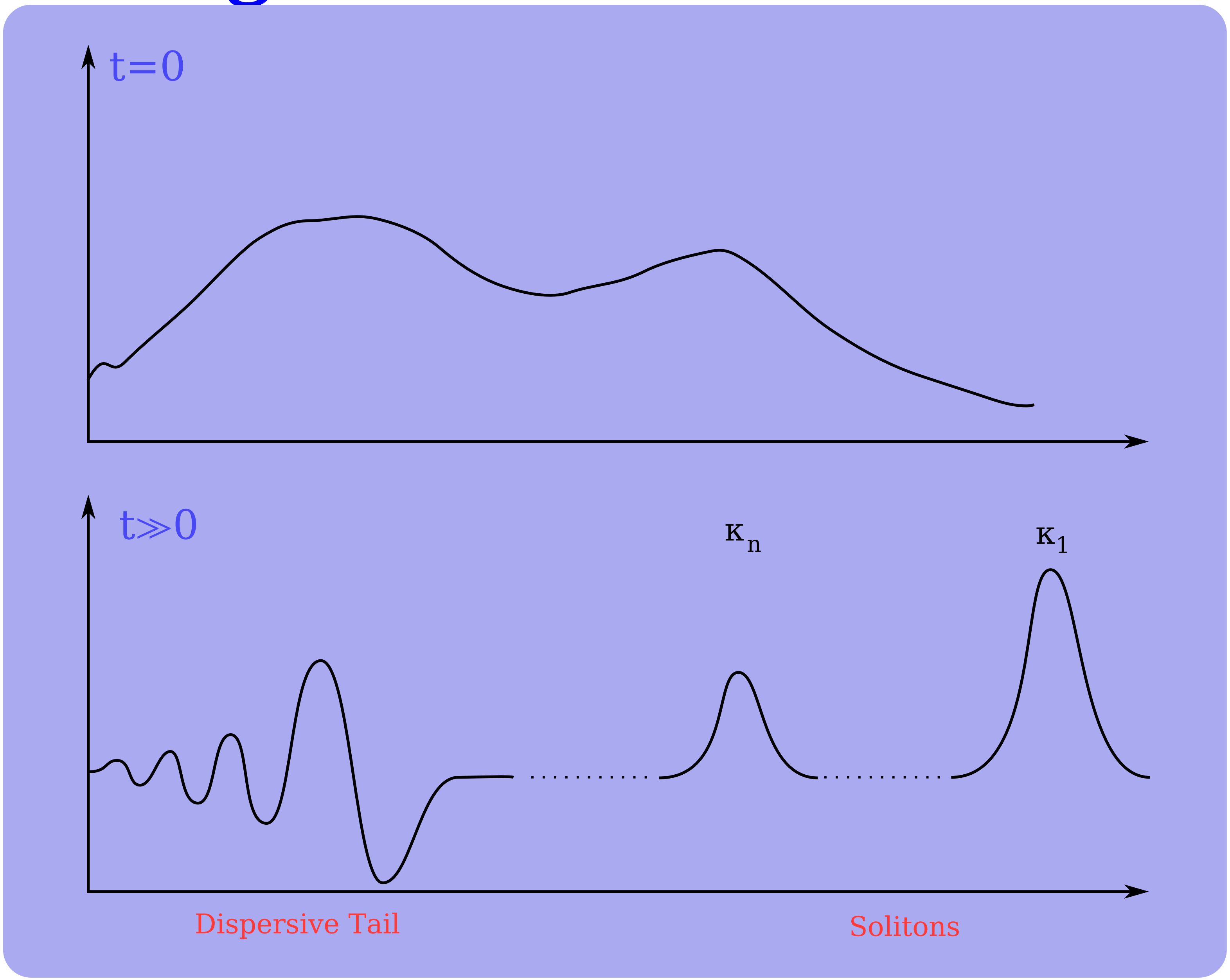


Solitons: speed and amplitude preserved upon interaction

Solitons: Two Soliton



Long Time Solution of KdV



The Dispersive Tail

- Description of the dispersive tail is related to equations studied by P. Painlevé (1893)
- Subsequent studies on many “soliton” equations find reduced equations with similar properties: Painlevé type equations
- Also find Chazy (1909) type equations which have different and more “exotic” properties than Painlevé equations
- Chazy’s equation connects with equations due to Ramanujan (1916) – MJA, S. Chakravarty, R. Halburd, H. Hahn (2003, 2006)

Painlevé

*

Water Wave Equations

Classical equations: Define the domain D by $D = \{-\infty < x, y < \infty, \quad -h < z < \eta(x,y,t), \quad t > 0\}$. The water wave equations satisfy the following system for $\phi(x,y,z,t)$ and $\eta(x,y,t)$:

$$\Delta\phi = 0 \ \text{ in } \ D,$$

$$\phi_z = 0 \ \text{ on } \ z = -h,$$

$$\eta_t + \nabla\phi \cdot \nabla\eta = \phi_z \ \text{ on } \ z = \eta,$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right), \ \text{ on } \ z = \eta,$$

where g: gravity, $\sigma = \frac{T}{\rho}$: T surface tension, ρ: density.

WW-Nonlocal Spectral Eq

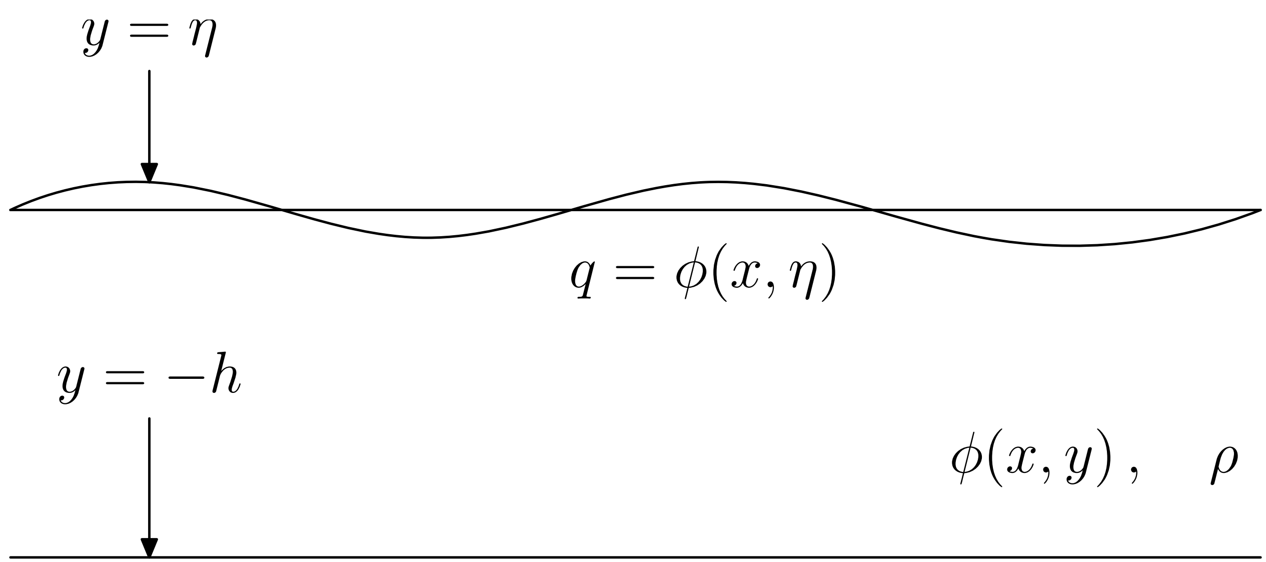
Work with A. Fokas, Z. Musslimani (JFM, 2006), reformulation: 2 eq., 2 unk: $\eta, q = \phi(x, \eta)$, rapid decay: 1 nonlocal spectral eq. and 1 PDE; fixed domain

$$\int dx e^{ik\cdot x}(i\eta_t \cosh[\kappa(\eta + h)] + \sinh\left[\kappa(\eta + h)\right] \frac{k \cdot \nabla q}{\kappa}) = 0 \qquad (I)$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} = \sigma \nabla \cdot (\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}) \qquad (II)$$

$$x = (x_1, x_2),\ k = (k_1, k_2),\ \kappa^2 = k_1^2 + k_2^2,\ q(x,t) = \phi(x, \eta(x,t))$$

WW: figure



Water wave configuration

Remarks

- Variables: η, q used by Zakharov ('68) in Hamiltonian formulation of WW
- Craig & Sulem ('93) derive Dirichlet-Neumann (DN) series in terms of η, q. Craig et al investigate WW and interfacial waves via DN operator methods

Remarks–con't

- From nonlocal formulation find:
  - Conserved quantities and new integral relations
  - Asymptotic reductions:
  - 1+1: KdV, Nonlinear Schrodinger (NLS) eq.
  - 2+1 Beneny-Luke and Kadomtsev-Petvashvili (KP) eq.
- MJA and Haut ('08,'09): nonlocal eqs for waves with 1 and 2 free interfaces; connect to DN operators/series, asymptotic reductions: high order asympt. expn's of 1-d and 2-d solitary waves

Nonlocal Eq: Dirichlet-Neumann Map

$$\int dx e^{ik\cdot x}\left(i\eta_t \cosh[\kappa(\eta+h)] + \sinh\left[\kappa(\eta+h)\right]\frac{k\cdot\nabla q}{\kappa}\right) = 0 \quad (I),$$

Take FT, exp'd cosh, sinh in pwrs of η find

$$\eta_t = G(\eta)q = \left(\sum_n A_n(\eta)\right)^{-1}\left(\sum_n B_n\,(\eta)\right)\,q$$

where $G(\eta)$ is the DN operator; series agrees with Craig et al; recall η_t prop. to normal deriv of ϕ

WW- Linearized System

If $|\eta|, |\nabla q|$ are small then eq. (I,II) simplify.

$$\int dx e^{ikx} (i\eta_t \cosh \kappa h + \frac{k \cdot \nabla q}{\kappa} \sinh \kappa h) = 0 \quad (1L)$$

recall $\kappa^2 = k_1^2 + k_2^2$. Use Fourier transform: $\hat{\eta} = \int dx e^{ikx} \eta$

$$i\hat{\eta}_t \cosh \kappa h + \frac{k \cdot \widehat{\nabla q}}{\kappa} \sinh \kappa h = 0 \quad (1L)$$

$$\widehat{q_t} + (g + \frac{\sigma}{\rho} \kappa^2)\hat{\eta} = 0 \quad (2L)$$

Then from eq. (1L), (2L)find:

$$\hat{\eta}_{tt} = -(g\kappa + \frac{\sigma}{\rho} \kappa^3) \tanh \kappa h \; \hat{\eta}$$

WW-Nonlocal System-Remarks

- Can find integral relations by taking $k \to 0$. First two (recall: $x \to (x_1, x_2)$)

$$\frac{\partial}{\partial t} \int dx \; \eta(x,t) = 0 \quad (Mass)$$

$$\frac{\partial}{\partial t} \int dx (x_1 \eta) = \int dx \; q_{x_1}(\eta + h) \quad (COM)$$

LHS: COM in x_1 direction -RHS related to x_1 momentum; COM -in x_2-direction and higher order virial identities can also be found.

Nondimensional Variables

We can make all variables nondimensional (nd):

$$x_1' = \frac{x_1}{l},\ x_2' = \gamma\frac{x_2}{l},\ t' = \frac{c_0}{l}t,\ a\eta' = \eta,\ q' = \frac{alg}{c_0}q,\ c_0 = \sqrt{gh}$$

where l, a are the characteristic wave length, height, and γ is a nd transverse length parameter. Eq are written in terms of

nd variables $\epsilon = \frac{a}{h} << 1$, small amplitude, $\mu = \frac{h}{l} << 1$ long waves; $\gamma << 1$, slow transverse variation; hereafter drop $'$

WW-Asymptotic Systems

Small ampl., long waves. with nd paramters:
$\epsilon = \frac{a}{h}, \mu = \frac{h}{l}; \epsilon, \mu, \gamma << 1$ Find: Benney-Luke (BL, 1964) eq. (nmlz'd surface tension, $\hat{\sigma} = \sigma - 1/3$):

$$q_{tt} - \tilde{\Delta} q + \hat{\sigma}\mu^2 \tilde{\Delta}^2 q + \epsilon(\partial_t |\tilde{\nabla} q|^2 + q_t \tilde{\Delta} q) = 0 \ (BL)$$

$$\tilde{\Delta} = \partial^2_{x_1} + \gamma^2 \partial^2_{x_2} \qquad |\tilde{\nabla} q|^2 = \left(q^2_{x_1} + \gamma^2 q^2_{x_2}\right).$$

If $\epsilon = \mu^2 = \gamma^2$ then BL yields KP equation; after rescaling KP eq in std form

$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3\,\mathrm{sgn}(\hat{\sigma}) u_{yy} = 0$$

WW-Asympt. Systems-con’t

- Small amplitude, long waves, slow transverse variation find 2+1d BL eq.; with maximal balance: $\epsilon = \mu^2 = \gamma^2 << 1$ find KP eq.
- Small amplitude, quasi-monchomatic wave exp’n for η, q and use the Riemann-Lesbesgue Lemma)–find nonlinear Schödinger (NLS) eq.
- Thus find both shallow and deep water reductions from nonlocal spectral problem
- Traveling wave solutions e.g. 1-d: $X = x_1 - c_X t$ or 2-d $X = x_1 - c_X t, Y = x_2 - C_Y t$; May find asymptotic series $\eta = \sum \epsilon^j \eta^j,\ q = \sum \epsilon^j q^j$ with max. balance, to high order in ϵ; η, q depend on surface tension.

Solitary waves–1d

Can find 1-d and 2-d solitary waves to high order (MJA and Haut (2009-2010). here 1-d: $X = x_1 - ct$:
$\eta(X) = \sum_{i=1}^{N} \sum_{j=1}^{i} \alpha^{2i} \epsilon^{i-1} d_{i-1,j} \text{sech}^{2j}(X\alpha)$
e.g. N=9 orders in ϵ. First two terms:

$$\begin{aligned}\eta &= -\frac{4}{3}\alpha^2(3\sigma - 1)\text{sech}^2(X\alpha) \\ &\epsilon\left(\frac{8}{9}\alpha^4\left(18\sigma^2 - 12\sigma + 1\right)\text{sech}^2(X\alpha) - \right. \\ &\left.\frac{4}{3}\alpha^4\left(12\sigma^2 - 4\sigma - 1\right)\text{sech}^4(X\alpha)\right) + O(\epsilon^2).\end{aligned}$$

We normalize $\alpha = \alpha(\epsilon)$ so that

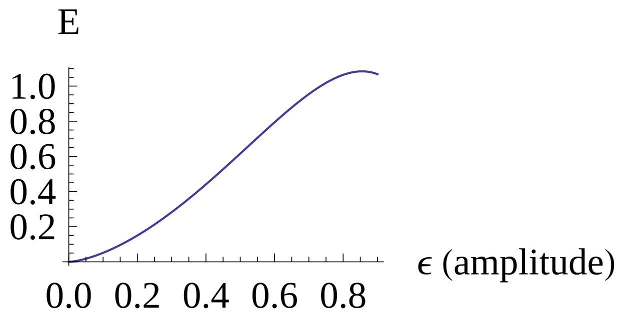
$$\eta(0) = \begin{cases} 1, & \text{if } 0 < \sigma < 1/3. \\ -1, & \text{if } \sigma > 1/3. \end{cases}$$

Solitary waves 1d: $\sigma < 1/3$

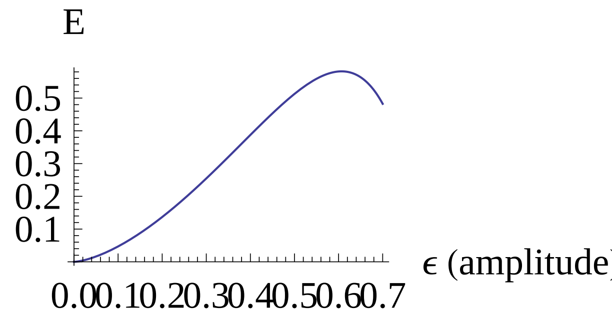
Find series for energy, $E = KE + PE + SE$ as function of ST:

i) $\sigma < 1/3$:

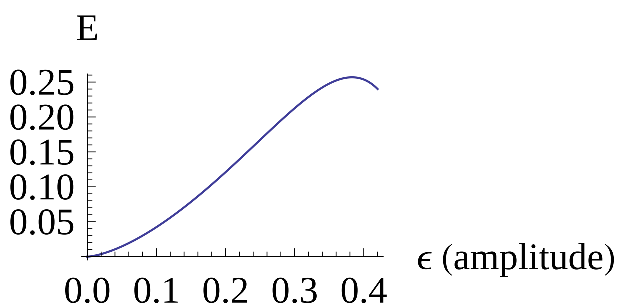
$\sigma = 0$



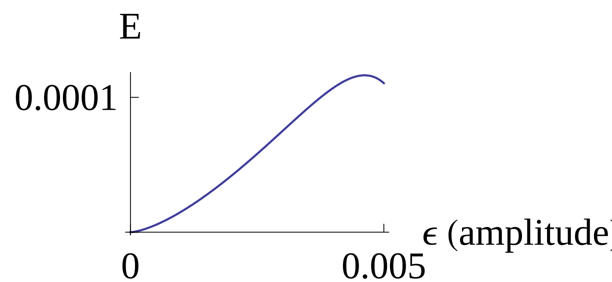
$\sigma = .05$



$\sigma = .1$



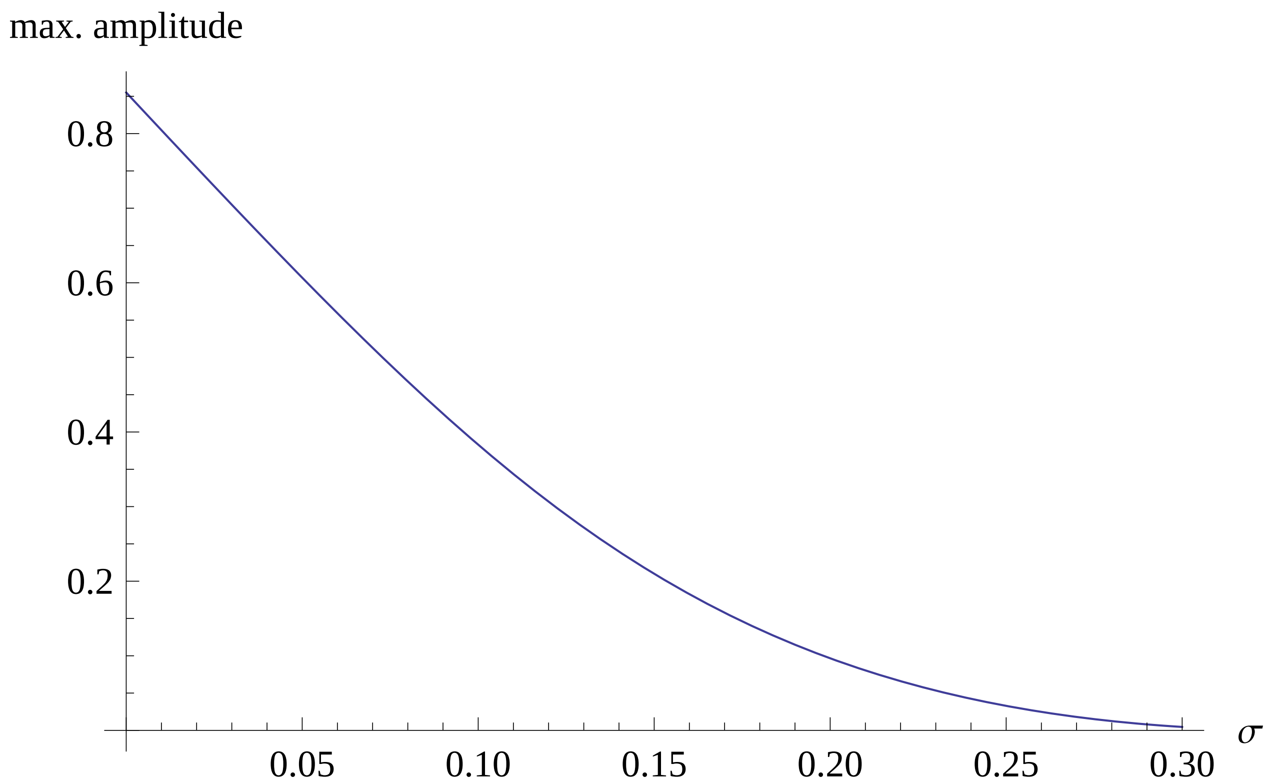
$\sigma = .3$



Plots of E, using the series with 9 terms, for given ST. Note max energy and amplitude decrease with ST

Solitary waves 1d: $\sigma < 1/3$–con't

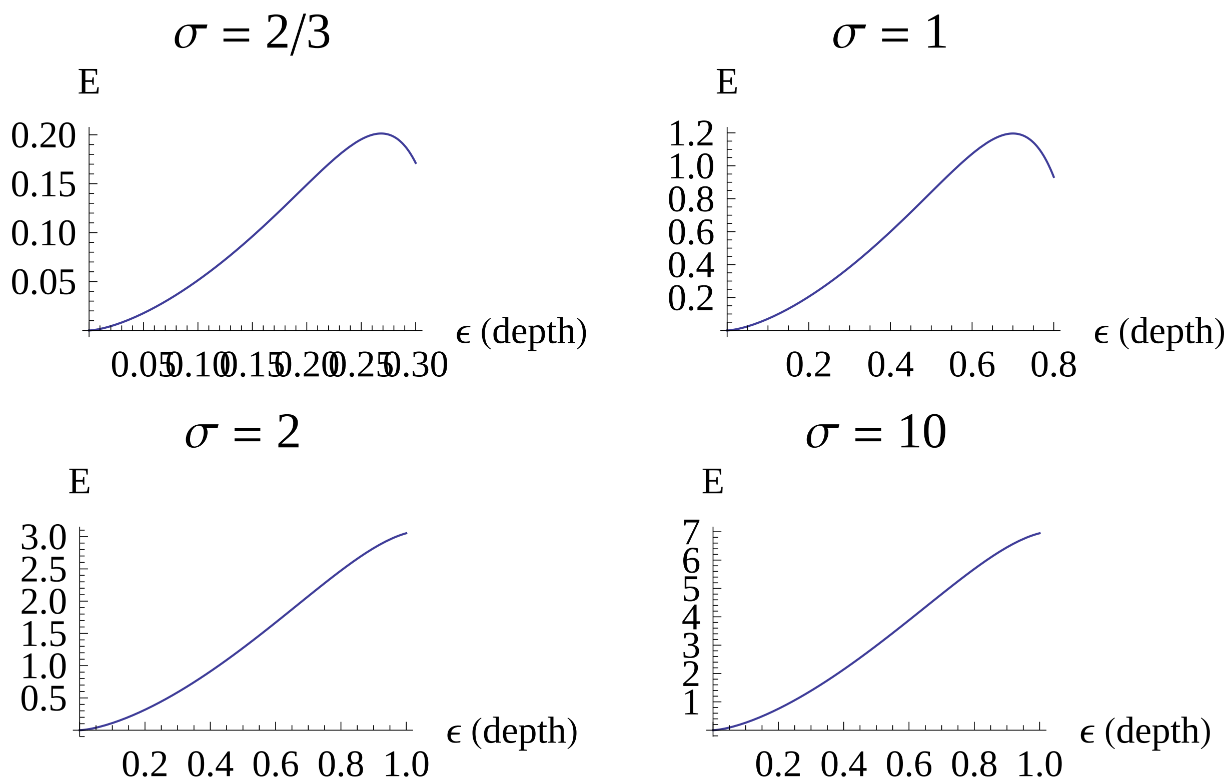
From $E = E(\sigma, \epsilon)$:



Plot of the amplitude of the most energetic wave as a function of ST ($\sigma < 1/3$); when $\sigma = 0$ results agree with Fenton and Longuet-Higgens (1974).

Solitary waves 1d: $\sigma > 1/3$

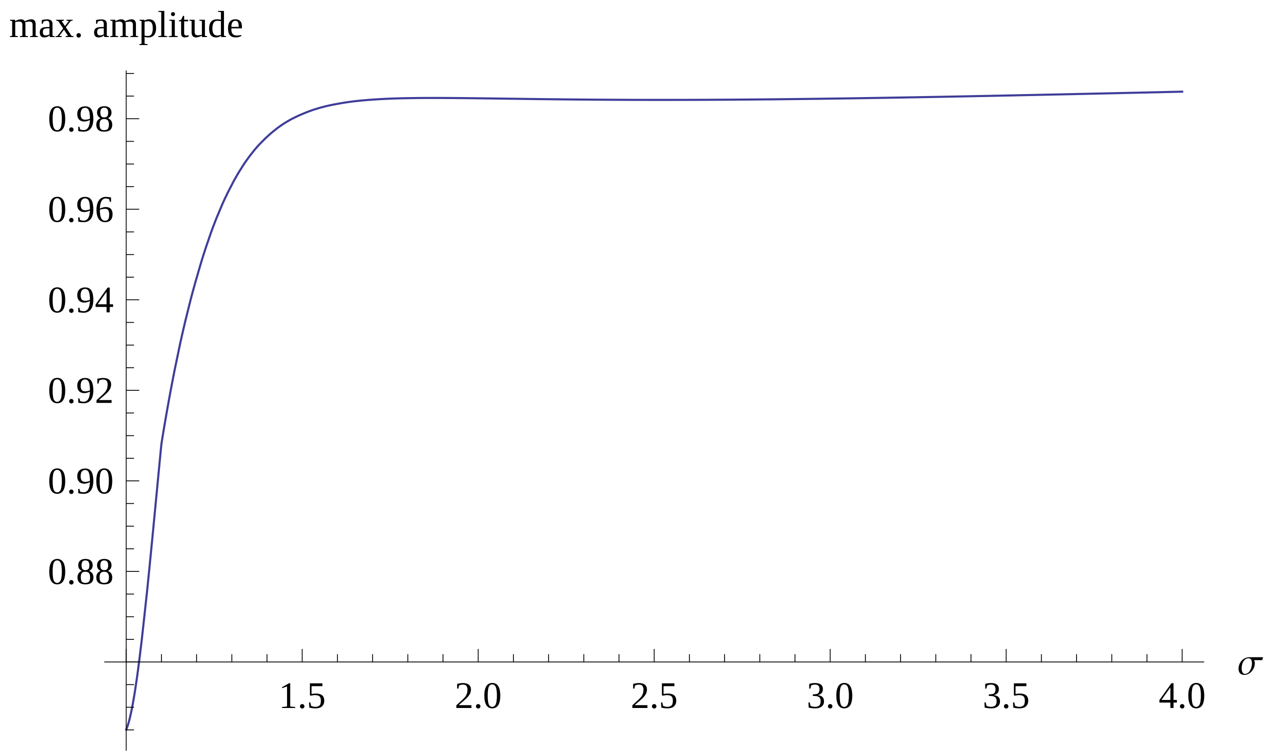
ii) Solitary waves of depression: $\sigma > 1/3$ –observed recently: Falcon (2002)



Energy E as a function of the (non-dimensional) fluid depth ϵ, for $\sigma = 2/3, 1, 2, 10$

Solitary waves 1d: $\sigma > 1/3$–con't

From $E = E(\sigma, \epsilon)$:



Amplitude of the most energetic wave as a function of ST, $\sigma > 1/3$; suggests that large ST solitary waves can get close to the bottom.

KP Equation

The KP equation in water waves (MJA & Segur 1979):

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{gh}}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \frac{h^2}{2}(\frac{1}{3} - \hat{T})\eta_{xxx}\right) + \frac{1}{2}\eta_{yy} = 0$$

KP equation in standard form

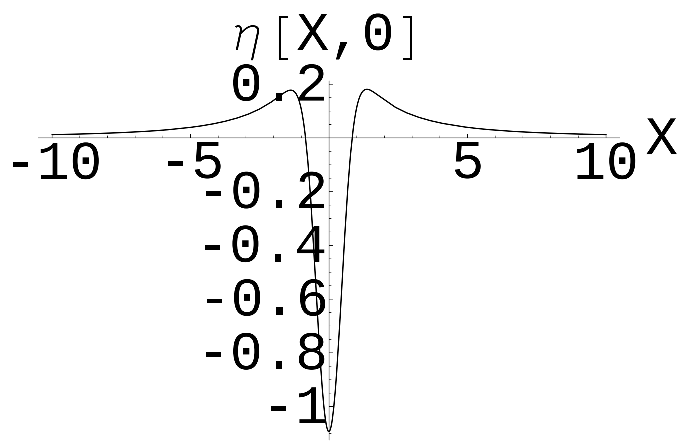
$$\partial_x(u_t + 6uu_x + u_{xxx}) + 3\,\mathrm{sgn}(\hat{\sigma})u_{yy} = 0$$

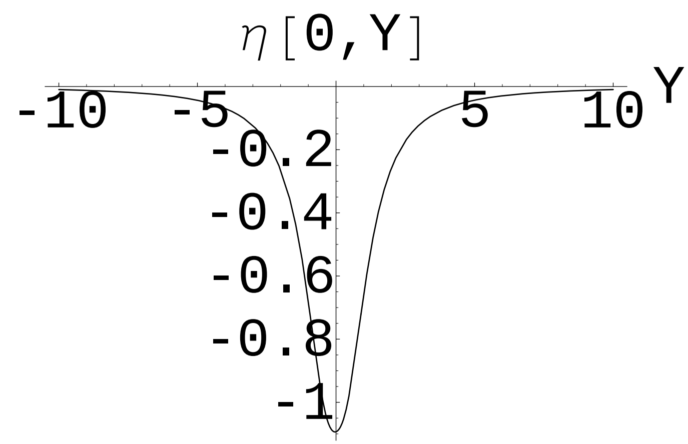
where $\hat{\sigma} = (\frac{1}{3} - \hat{T}), \hat{T} = \frac{T}{\rho g h^2}, \quad \rho$ = density

Solitary waves –additional remarks

2-d solitary waves can be computed by similar procedure as 1-d. Leading order solution is the lump solution of KP: $\hat{\sigma} > 1/3$. Higher order perturbation results are only numerical.

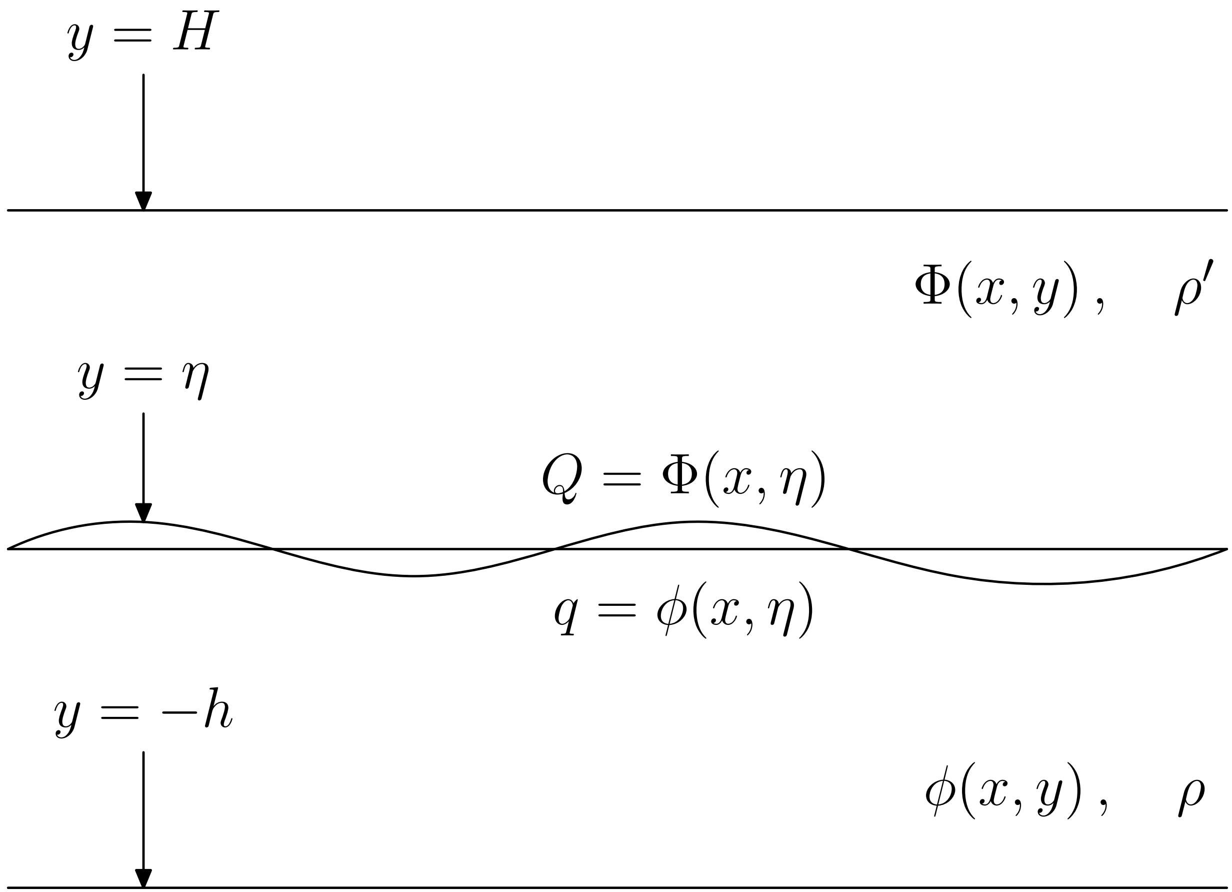
Typical wave profiles at $Y = 0$ and $X = 0$ below.





Plots of $\eta(X, 0)$ and $\eta(0, Y)$, for $\epsilon = 0.56$, and $1 - c^2 = 0.47$

Interfacial wave–rigid top (RT)



Interfacial wave-RT–nonlocal form

$$\int_{\mathbf{R}^2} e^{ikx}\cosh(\kappa(\eta+h))\eta_t\,dx = i\int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa(\eta+h))\left(\frac{k}{\kappa}\cdot\nabla q\right)dx$$

$$\int_{\mathbf{R}^2} e^{ikx}\cosh(\kappa(\eta-H))\eta_t\,dx = i\int_{\mathbf{R}^2} e^{ikx}\sinh(\kappa(\eta-H))\left(\frac{k}{\kappa}\cdot\nabla Q\right)dx$$

$$\rho\left(q_t+\frac{1}{2}|\nabla q|^2+g\eta-\frac{(\eta_t+\nabla q\cdot\nabla\eta)^2}{2\,(1+|\nabla\eta|^2)}\right)-$$

$$\rho'\left(Q_t+\frac{1}{2}|\nabla Q|^2+g\eta-\frac{(\eta_t+\nabla Q\cdot\nabla\eta)^2}{2\,(1+|\nabla\eta|^2)}\right)=\sigma\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)$$

where quantities defined in prior figure-and earlier.; 3 eq., 3 unkowns η, q, Q: fixed domain!

Interfacial wave-RT–remarks

Small ampl., long waves- find ILW-BL eq (nondim):

$$Q_{tt} - c_0^2 Q_{x_1x_1} - \gamma^2 c_0^2 Q_{x_2x_2} + \epsilon\left(2Q_{x_1}Q_{x_1t} + Q_t Q_{x_1x_1}\right) +$$
$$\mu\left(\frac{\tilde{\sigma}}{\tilde{\rho}} Q_{x_1x_1x_1x_1} - \frac{c_0^2}{\tilde{\rho}} i \coth(\alpha D_1) Q_{x_1x_1x_1}\right) = 0\,,$$

where $\mu = \frac{H}{l_x}, \epsilon = \frac{a}{H}, \gamma$ as in std BL, $\tilde{\rho} = \frac{\rho'}{\rho}, \delta = \frac{h}{H}, \alpha = \delta\mu$, $c_0^2 = (1/\tilde{\rho} - 1)$, and psd notation ($D_1 = -i\partial_{x_1}$):

$$\coth(\alpha D_1) Q_{x_1x_1x_1} = \frac{1}{(2\pi)^2 i} \int_{\mathbf{R}^2} e^{ikx} \coth(\alpha k_1) k_1^3 \widehat{Q}(k)\, dk\,.$$

Interfacial wave-RT–ILW-KP

Reduce to one direction: $x = x_1 - c_0 t,\ y = x_2, T = \epsilon t,\ w = Q_x,$ yields ILW-KP

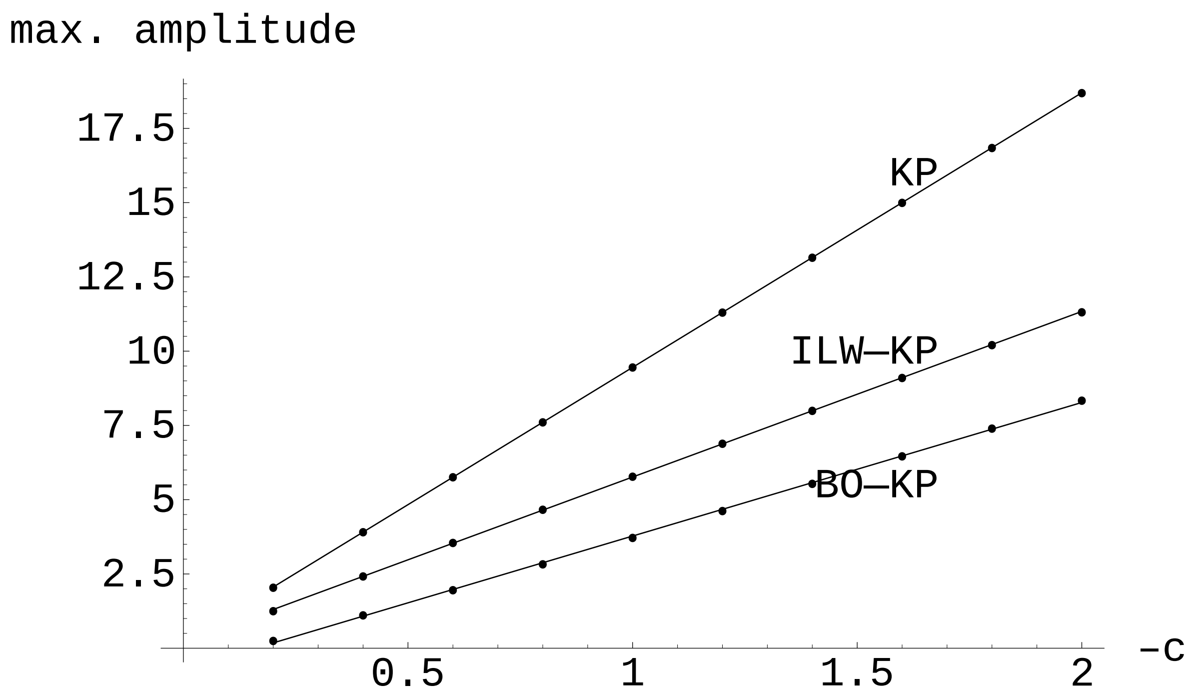
$$2c_0 w_{xt} + c_0^2 w_{yy} + \frac{3}{2} c_0 \partial_x^2 \left(w^2\right) - \frac{\tilde{\sigma}}{\tilde{\rho}} w_{xxxx} + \frac{ic_0^2}{\tilde{\rho}} \coth(\alpha D_1) w_{xxx} = 0\,,$$

Recall: $\delta = \frac{h}{H}, \alpha = \delta\mu$; if $\alpha \to 0$ find standard KP eq.; If $\alpha \to \infty$ find BO-KP eq. (cf. MJA, Segur '80)

$$2c_0 w_{xt} + c_0^2 w_{yy} + \frac{3}{2} c_0 \partial_x^2 \left(w^2\right) - \frac{\tilde{\sigma}}{\tilde{\rho}} w_{xxxx} - \frac{c_0^2}{\tilde{\rho}} H w_{xxx} = 0\,,$$

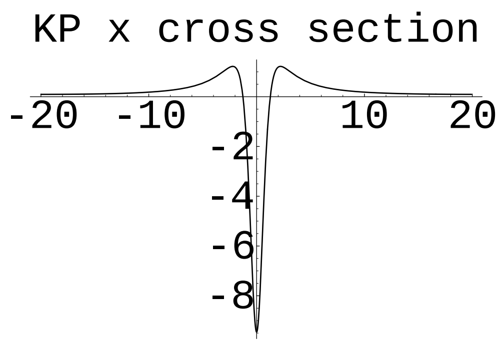
where $Hu(x) = P\frac{1}{\pi} \int \frac{u(\xi)}{\xi - x}\, d\xi$

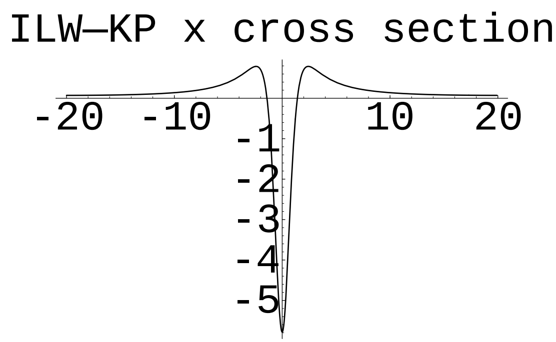
ILW-KP Lumps

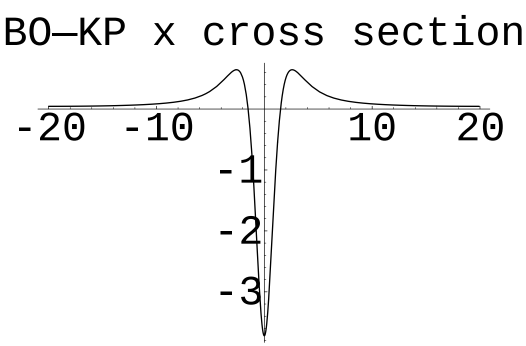


Max. amp. vs x-velocity c for $\alpha = 1/10, 1, 10$ i.e. KP. ILW-KP, and BO-KP resp.

ILW-KP Lumps-con’t

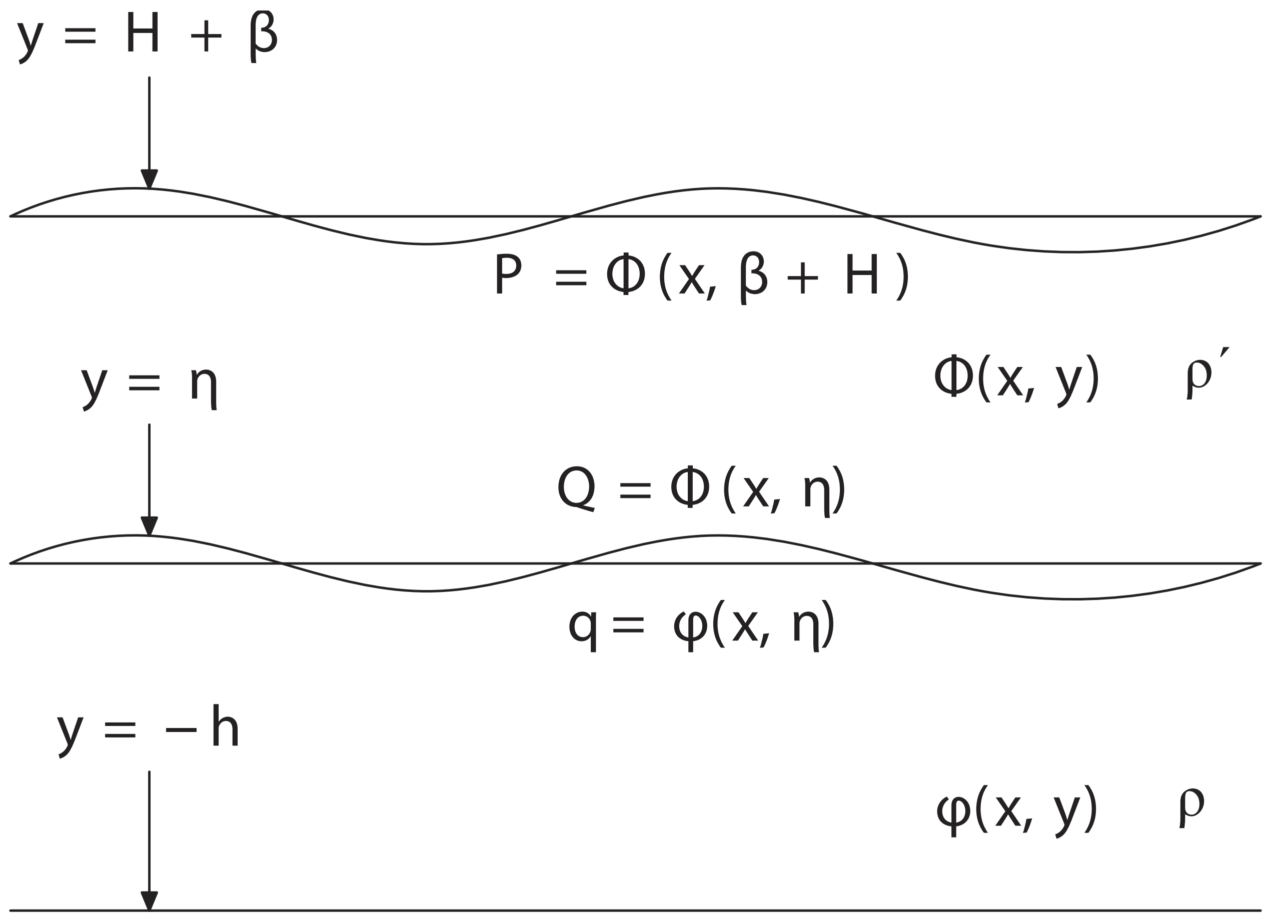






Typical x cross sections $w(x,0)$, $\alpha = 1/10, 1, 10$ i.e. KP. ILW-KP, and BO-KP resp. with $c = c_x = -1, c_y = 0$

Interfacial wave and free surface(FS)



Two-fluid configuration with two free interfaces

Interfacial waves-FS–nonlocal form

$$\int_{\mathbf{R}^2} e^{ikx} \cosh(\kappa(\eta + h)\eta_t \, dx = i \int_{\mathbf{R}^2} e^{ikx} \frac{\sinh(\kappa(\eta + h)}{\kappa} (k \cdot \nabla) q\} \, dx$$

$$\int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa\beta)\beta_t \, dx - \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\eta - H))\eta_t \, dx =$$

$$- i \int_{\mathbf{R}^2} e^{ikx} \frac{\cosh(\kappa(\eta - H))}{\kappa} (k \cdot \nabla) \, Q \, dx + i \int_{\mathbf{R}^2} e^{ikx} \frac{\cosh(\kappa\beta)}{\kappa} (k \cdot \nabla) \, P \, dx$$

$$\int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa(\beta + H))\beta_t \, dx - \int_{\mathbf{R}^2} e^{ikx} \sinh(\kappa\eta)\eta_t \, dx =$$

$$- i \int_{\mathbf{R}^2} e^{ikx} \frac{\cosh(\kappa\eta)}{\kappa} (k \cdot \nabla) \, Q \, dx + i \int_{\mathbf{R}^2} e^{ikx} \frac{\cosh(\kappa(\beta + H))}{\kappa} (k \cdot \nabla) \, P \, dx$$

Interfacial waves-FS–Bernoulli Eq.

$$\rho\left(q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q\cdot\nabla\eta)^2}{2\left(1+|\nabla\eta|^2\right)}\right)$$

$$-\rho'\left(Q_t + \frac{1}{2}|\nabla Q|^2 + g\eta - \frac{(\eta_t + \nabla Q\cdot\nabla\eta)^2}{2\left(1+|\nabla\eta|^2\right)}\right) = \sigma_1\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right)$$

$$P_t + \frac{1}{2}|\nabla P|^2 + g\beta - \frac{(\beta_t + \nabla P\cdot\nabla\beta)^2}{2\left(1+|\nabla\beta|^2\right)} = \frac{\sigma_2}{\rho'}\nabla\cdot\left(\frac{\nabla\beta}{\sqrt{1+|\nabla\beta|^2}}\right)$$

Thus have 5 eq. in 5 unkowns: η, β, q, Q, P: fixed domain!

Interfacial waves-FS–Remarks

- Can find integral relations by taking $k \to 0$; e.g.

$$\frac{\partial}{\partial t} \int dx\ (\eta(x,t) - \beta(x,t)) = 0 \quad (Mass)$$

- One interface with RT is a special subcase
- Can find asymptotic reductions; Long waves: ILW-BL and ILW-KP. May also find coupled NLS eq.: small amplitude, quasi-monochomatic limit

Interfacial and FS waves–Coupled NLS

Eg let

$$\eta(x,y,t) = e^{i(k_1x+l_1y-\omega_1t)}\eta_{1,0}(X,T) + e^{i(k_2x-\omega_2t)}\eta_{0,1}(X,T) + c.c. + O(\epsilon)\,,$$

$$\beta(x,y,t) = e^{i(k_1x_1+l_1y-\omega_1t)}\beta_{1,0}(X,T) + e^{i(k_2x-\omega_2t)}\beta_{0,1}(X,T) + c.c. + O(\epsilon)\,,$$

where $X = \epsilon x$, $T = \epsilon t$. Find coupled NLS

$$if_\tau + \delta_1 f_{\xi\xi} + (|f|^2 + \delta_2|g|^2)f = 0\,,$$

$$ig_\tau + i\Delta g_\xi + \delta_3 g_{\xi\xi} + (\delta_4|f|^2 + |g|^2)g = 0\,,$$

where $\xi = X - C_{g_1}T$, $\tau = \epsilon T$; the coef. $\delta_1,\ldots,\delta_4$ depend on k_1, l_1, k_2, ω_1, ω_2, and fluid parameters, $\delta_1 = \omega_{1,k_1k_1}/2, \delta_3 = \omega_{2,k_2k_2}/2, f(\xi,\tau) = \eta_{1,0}(X,T), g(\xi,\tau) = \beta_{0,1}(X,T), C_{g_2} = C_{g_1} + \epsilon\Delta$

See MJA & Haut ’09

Nonlinear optics

In nonlinear optics localized waves are often termed solitons.
Some problems of interest

- Fiber optic communications
- Ultra-short pulses in mode-locked lasers
- Photonic Lattices
- In nonlinear optics nonlinear Schrödinger (NLS) equations play an important role

Fiber Optic Communications

- In 1973 researchers found that a well-known soliton equation (NLS) models waves in optical fibers and solitons proposed for communications–Hasgawa and Tappert (1973)
- Experiments –Molleneauer et al 1980’s
- Dispersion Management (DM) reduces “penalties”
  - DM commercial: 2001-present
  - DM pulses: solitons, quasi-linear ...
- Mathematical methods to analyze DM systems; e.g. MJA & G. Biondini (1998), MJA, T. Hirooka, A. Docherty, C. Ahrens (2002-2006)...

Dispersion-Managed System

Normalized eq.

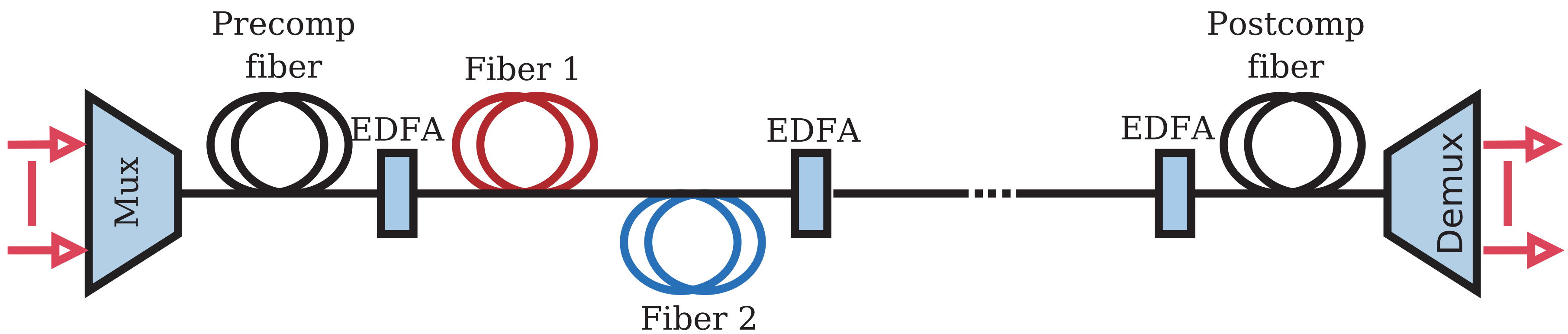
$$iu_z(z,t) + \frac{1}{2}d(z)u_{tt} + n(z)|u|^2u = 0$$

- $d(z) =< d > +\frac{1}{z_a}\Delta\left(\frac{z}{z_a}\right), \quad 0 < z_a \ll 1,$
- $\Delta = \begin{cases} \Delta_1 & \text{in “anomalous”} \\ \Delta_2 & \text{in “normal”} \end{cases}$

  $n(z) = n\left(\frac{z}{z_a}\right)$: damping- amplification; lossless: $n = 1$
- $\langle\Delta\rangle = \frac{\int_0^{z_a}\Delta dz}{z_a} = 0$

  Typical numbers: Power= 1mW; pulse width= 10ps=10^{-11}s; amplifier length=40km; find $z_a = 0.1$

Dispersion Management



DM used in current undersea transmission systems

Asymptotic analysis

Multiple scales (ref. MJA & G. Biondini Opt.Lett. '98)

$$u = u(\zeta, Z, t; z_a) : \zeta = \frac{z}{z_a}, Z = z; \quad 0 < z_a \ll 1$$

$\partial_z u = \frac{1}{z_a}\partial_\zeta u + \partial_Z u; \quad u = u^{(0)} + z_a u^{(1)} + \ldots$

Take Fourier transform (FT) find:

$$\hat{u}^{(0)} = \hat{U}(Z, \omega) e^{-\frac{i}{2}\omega^2 C(\zeta)}, \quad C(\zeta) \equiv \int_0^\zeta \Delta(\zeta') d\zeta'$$

Next order, secularity condition determines $\hat{U}(Z, \omega)$: satisfies the **DMNLS** equation which is nonlinear & nonlocal (see also Gabitov & Turitsyn '96)

DMNLS equation

$$i\hat{U}_Z - \frac{1}{2}\omega^2 < d > \hat{U} + < n e^{\frac{i}{2}\omega^2 C}\mathcal{F}\{|u^{(0)}|^2 u^{(0)}\} >= 0 \quad \text{(DMNLS)}$$

where $< F >= \int_0^1 F d\zeta;$ also find $< \cdot >$ can be written:
$< \cdot >= \iint r(\omega_1\omega_2)\hat{U}(\omega+\omega_1)\hat{U}(\omega+\omega_2)\hat{U}^*(\omega+\omega_1+\omega_2)d\omega_1 d\omega_2$
where $r(x)(2\pi)^2 =< n e^{iCx} >$;

If $\Delta \to 0 \Rightarrow C \to 0$ find NLS eq. in Fourier domain.

Solitons: $\hat{U}(z,\omega) = \hat{f}(\omega)e^{i\lambda^2 z/2}$; yield fixed point integral eq; use mode finding algorithms or further approximations–strong DM soltions have approx. gaussian envelope and they "breathe"

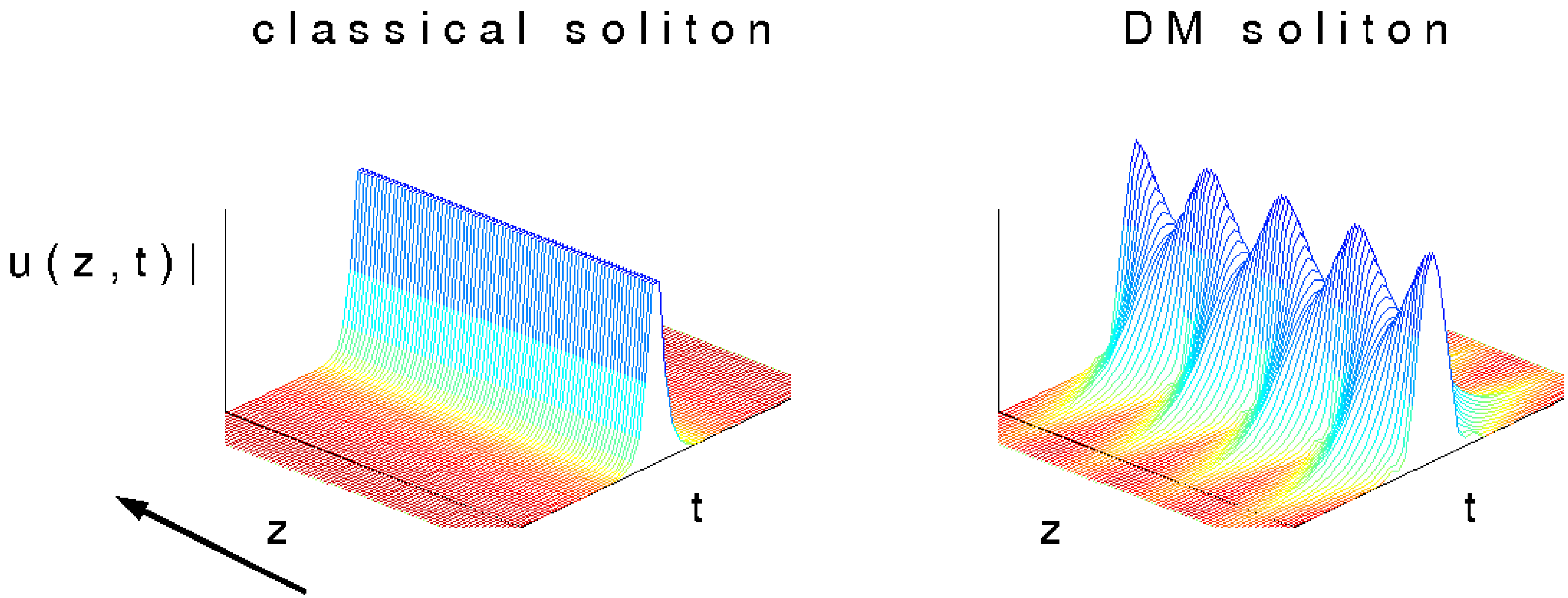
DM Solitons

DM solitons, ansatz: $\hat{U}(z,\omega) = \hat{f}(\omega)e^{i\lambda^2 z/2}$,

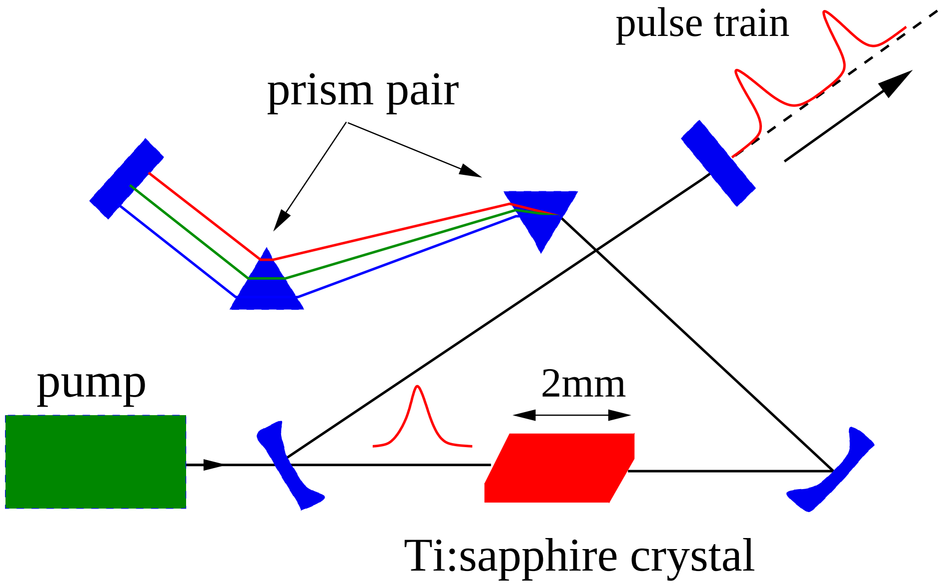
$$-\frac{\lambda^2}{2}\hat{f} - \frac{\omega^2}{2} < d > \hat{f} + \iint r(\omega_1\omega_2)\hat{f}\cdot\hat{f}\cdot\hat{f}^* \, d\omega_1 d\omega_2 = 0$$

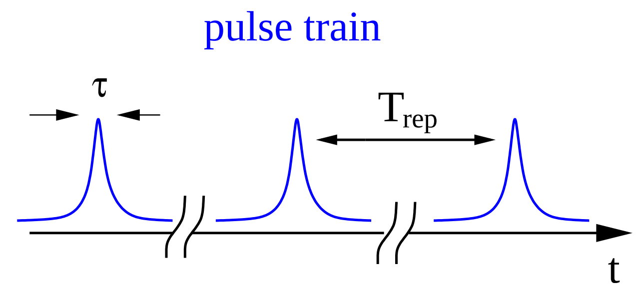
- Nonlinear fixed-point eq. – numerical computations $< d >> 0$; Petviashvili method (1976), MJA, Musslimani (OL, 2005); Existence: Zharnitsky et al (PRE, 2000)
- When $C = 0$ recover classical case: $f(t) = \lambda\,\mathrm{sech}(\lambda t)$.
- Can also find: higher-order DMNLS equation and multi-humped DM solitons (MJA,T. Hirooka,T. Inoue, JOSAB 2002)

DM: Breathing Soliton



DM Solitons: Mode-locked lasers





$$\tau = 10\text{fs} = 10^{-14}\,\text{sec}, \quad T_{\text{rep}} = 10\text{ns} = 10^{-8}\text{s},$$

Experiments (S. Cundiff's lab U. CO.)/theory: DM solitons 2004-05. Applications: highly stable oscillators, optical clocks...; Improved theories: MJA, T. Horikis, B.Ilan, S. Nixon, Y. Zhu 2008–

Power-Energy-Saturation Eq.

Lasers: power-energy saturation (PES) *distributed* eq. normalized

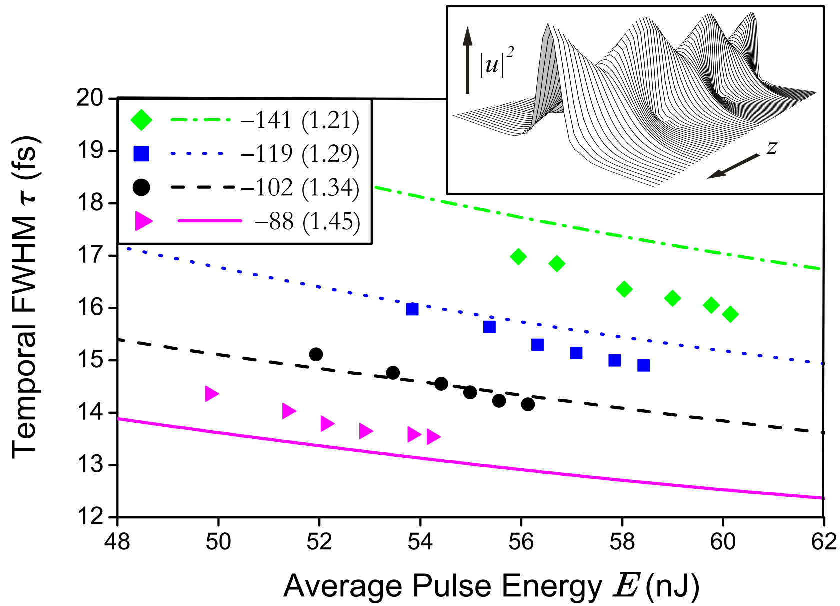
$$iu_z + \frac{d(z)}{2}u_{tt} + n(z)|u|^2u = R[u] =$$

$$\frac{ig}{1+E/E_0}u + \frac{i\tau}{1+E/E_0}u_{tt} - \frac{il}{1+P/P_0}u$$

with $E(z) = \int_{-\infty}^{+\infty} |u|^2 dt, P(z,t) = |u|^2$, $g, \tau, l, E_0, P_0 > 0$ const.,
RHS: NL gain, filtering, saturable loss, resp.

- Originally we assumed DM but no gain-filtering-loss: $g = \tau = l = 0$;
- We have studied PES eq. extensively

Theory vs. Ti:sapphire Experiments



Curves: DMNLS Symbols: experiments

GDD = $\langle k'' \rangle l_c$ (in fs^2)

- Good agreement from DMNLS; only one fitting parameter for all GDDs
- Pulses are well approximated by DM solitons (PRL, '05)

PES and “Master” Laser Equation

- “Master” laser equation obtained by Taylor series of PES–cubic nonlinearity.
- First proposed by Haus (1970’s–90’s) subsequently studied by numerous researchers. Reviews: H. Haus 2000, J.N. Kutz 2006.
- Only a small range of parameters for which master eq. yields mode-locking to localized pulses; i.e. mode-locking to solitons.

PES Eq. –con't

Distributed eq. (PES):

$$iu_z + \frac{d(z)}{2}u_{tt} + n(z)|u|^2u = R[u] =$$
$$\frac{ig}{1+E/E_0}u + \frac{i\tau}{1+E/E_0}u_{tt} - \frac{il}{1+P/P_0}u$$

with $E(z) = \int_{-\infty}^{+\infty} |u|^2 dt, P(z,t) = |u|^2$, $g, \tau, l, E_0, P_0 > 0$ const.
RHS: NL gain, filtering, saturable loss; fast sat absorber:
$A = \frac{l}{1+P/P_0}$

Haus (1975); Ilday et al (2004), Chong et al (2008) transfer function related to A in lumped model; Kaertner group (2009): compared ti:s lasers. PES eq. using distributed A–not previously studied in depth.

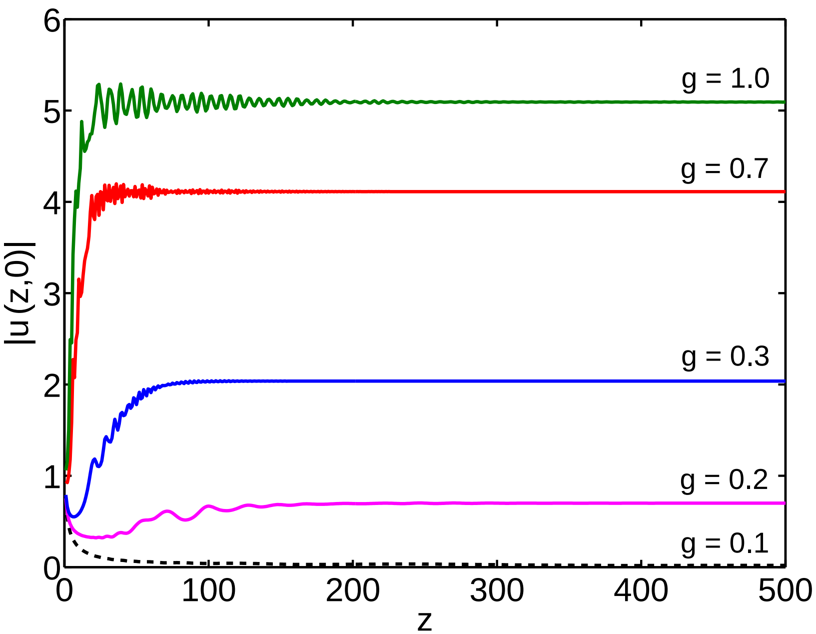
PES– Const. Dispersion

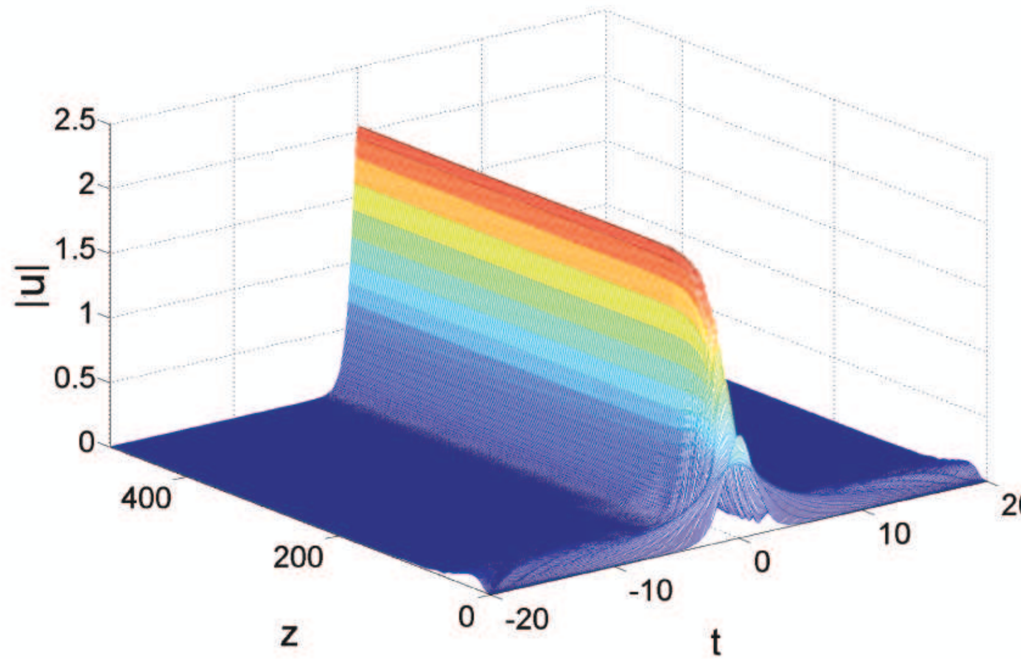
Typical situation (also true for exp'tl numbers):
$E_0 = P_0 = 1, d(z) = n(z) = 1 > 0$ (anomalous dispersion);
$\tau = l = 0.1$ and modify the gain parameter g. DM is similar.

First evolution from general state: unit gaussian IC

PES– Const. D–con't

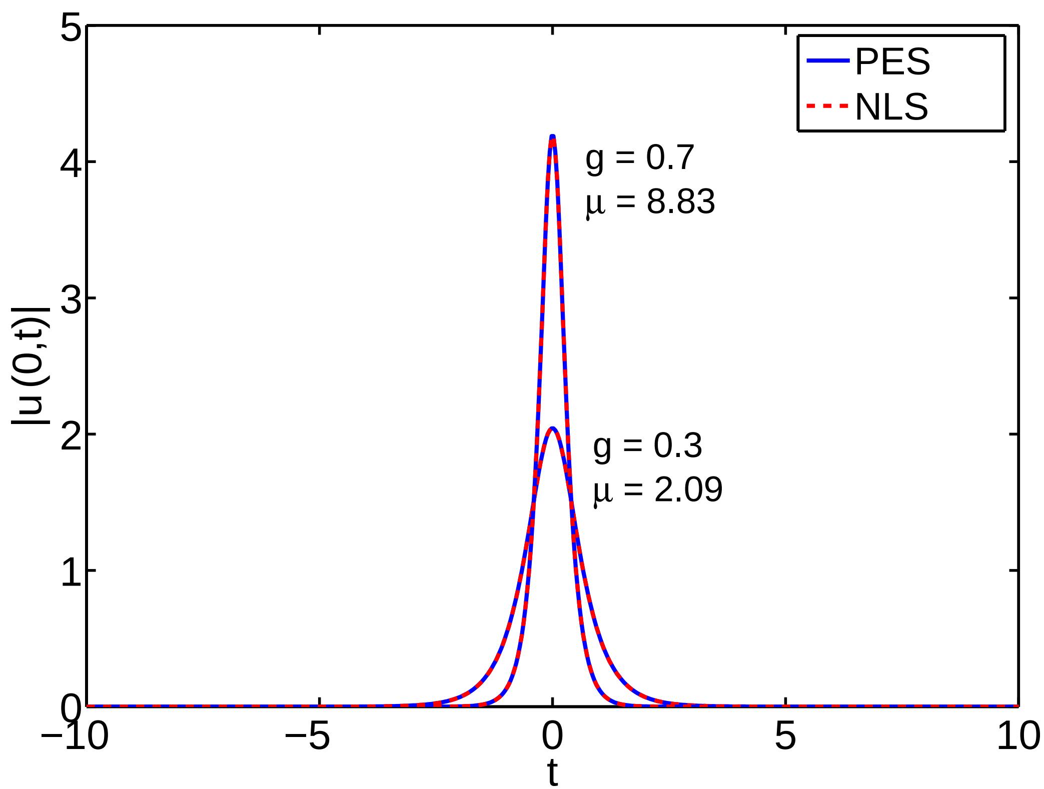
Left: Evolution of the pulse peak of a gaussian initial profile for PES eq. with different values of gain; damped pulse-peak evolution given with a dashed line. Right: evolution for $g = 0.3$.





PES– Const. D Solitons

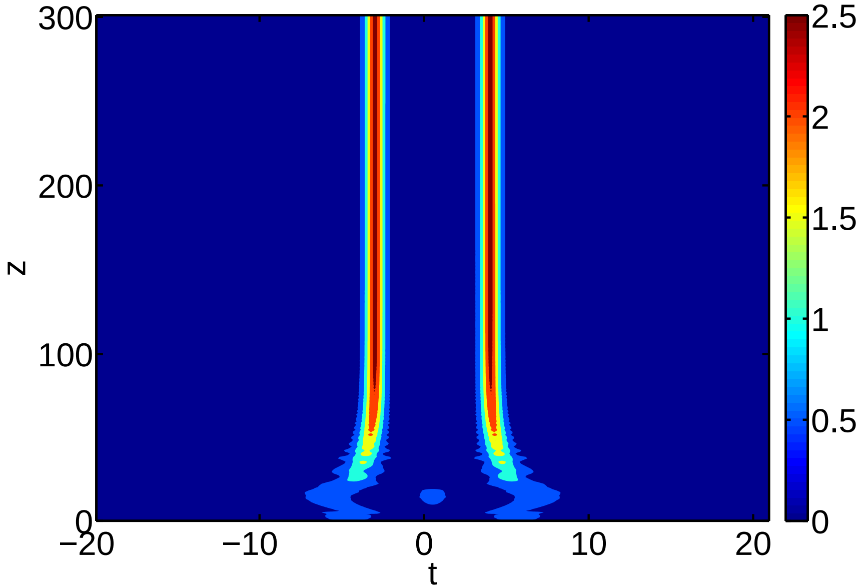
Soliton solutions of form: $u(z,t) \to u(0,t)e^{i\mu z}$, are well approximated by classical NLS solitons & correspond to specific values of μ, i.e. mode-locking–in contrast to classical NLS where solutions exist for $\mu > 0$.

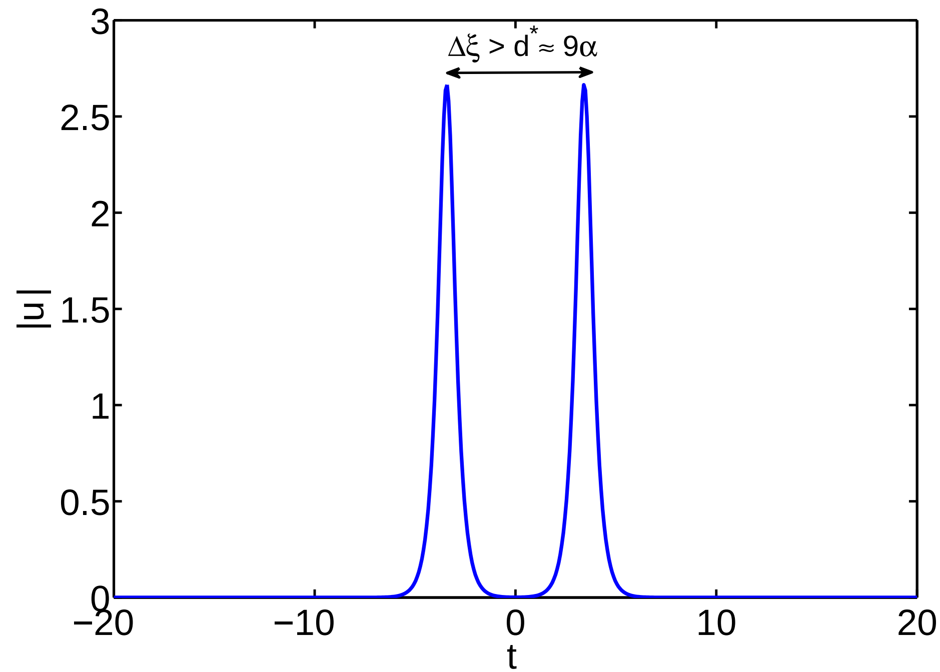


Above: typical soliton solutions of the NLS and PES eq.

PES-Const. D Soliton Strings

For PES eq. can also find soliton strings





Left: typical mode-locking evolution from two unit gaussians to an "in-phase bi-soliton" when $\Delta\xi > d^*$

Right: resulting soliton profile at $z = 300$, with $g = 0.5$

Analysis supports this observation; may find larger number of solitons/string

PES & DM

PSE with dispersion-management (DM):

$$iu_z + \frac{d(z)}{2}u_{tt} + n(z)|u|^2u = R[u] =$$
$$\frac{ig}{1+E/E_0}u + \frac{i\tau}{1+E/E_0}u_{tt} - \frac{il}{1+P/P_0}u$$

where $E = \int_{-\infty}^{+\infty} |u|^2 \,\mathrm{d}t;\;\; P = |u|^2,\;\; g, \tau, l > 0; , E_0, = P_0 = 1, d(z)$
is large and both $d(z), n(z)$ are rapidly varying (r.v.)

$$d(z) = d_0 + \frac{\Delta(z/z_a)}{z_a}, \quad n(z) = n(z/z_a), \;\; 0 < z_a \ll 1$$

Note: if $d(z)$ is r.v. and $O(1)$ then, similar to communications, multi-scale averaging theory can be applied

Mode-Locked DM Lasers

Typical two-step map: NL propagation only inside the active media–e.g. Ti:S crystal: $\Delta < 0, n = 1$ inside crystal; $\Delta > 0, n = 0$ elsewhere (linear)

DM Analysis:

- Multiple scales: $\zeta = \dfrac{z}{z_a}$ (short), $Z = z$ (long)
- Expand: $u = u^{(0)} + z_a u^{(1)} + \cdots, \quad 0 < z_a \ll 1$
- Solve the linear equation at $O(z_a^{-1})$ by FT–in t.
- Use the solution, $u^{(0)}$, to *remove* secularities at the $O(1)$ equation to derive the DM equation with $gain\text{-}loss$ terms: DM-PES
- Map strength parameter: $s = \dfrac{\Delta_1}{2}$

DM-PES Eq.

DM-PES eq. ("averaged eq.") has the form

$$i\frac{\partial \hat{U}_0}{\partial Z} - \frac{d_0}{2}\omega^2 \hat{U}_0 +$$

$$\int_0^1 \exp\left[i\frac{\omega^2}{2}C(\zeta)\right]\left(\mathcal{F}\left\{n(\zeta)|u^{(0)}|^2 u^{(0)} - R[u^{(0)}]\right\}\right)\, \mathrm{d}\zeta = 0$$

where

$$\hat{u}^{(0)} = \hat{U}_0(Z,\omega) exp\left[-i\frac{\omega^2}{2}C(\zeta)\right], \quad \mathrm{C}(\zeta) = \int_0^\zeta \Delta(\zeta')\, \mathrm{d}\zeta'$$

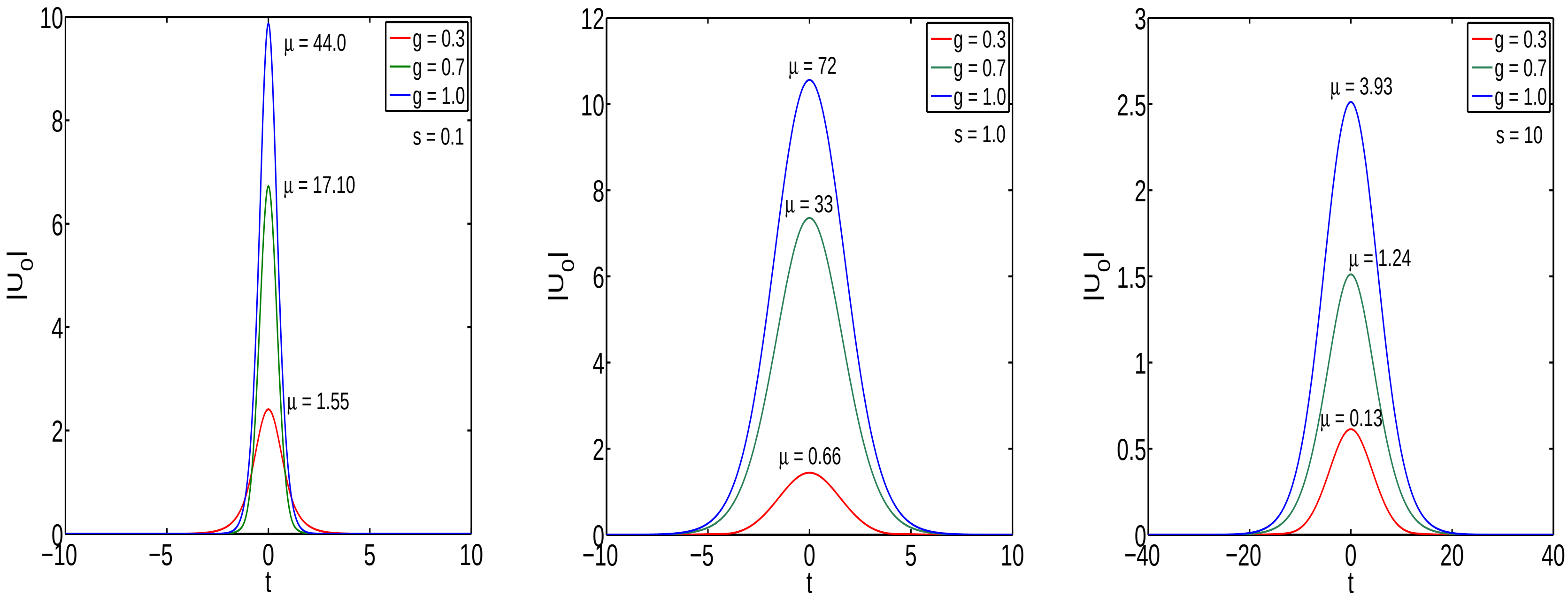
This is a *nonlocal* eq. for $\hat{U}_0(Z,\omega)$ and describes the *averaged* dynamics of the pulse envelope.

Averaged Eq. – Properties

- For net dispersion $d_0 > 0$ averaged eq. has mode-locking for a *wide range* of parameters
- Given map strength, s, find for $g < g_*$: pulses damp, $g > g_*$: mode-locking; where $g_* = g_*(s)$
- Typical: fix $\tau = l = 0.1$; vary s and the gain parameter g
- We also separately find solitons, specific values of μ for $g > g_*(s)$; $U_0(Z, t) \rightarrow U_0(t)e^{i\mu Z}$ These solitons correspond to the mode-locked states.
- Once mode-locked the solitons of the DM-PES eq. are well approximated by those of DMNLS

Ave. Eq. – Solitons

Solitons of the DM-PES equation corresponding to different map strengths and gain parameters:



Solitons correspond to specific values of μ, i.e. mode-locking.
All solutions are well-approximated by DMNLS solitons

Normal dispersion–PES

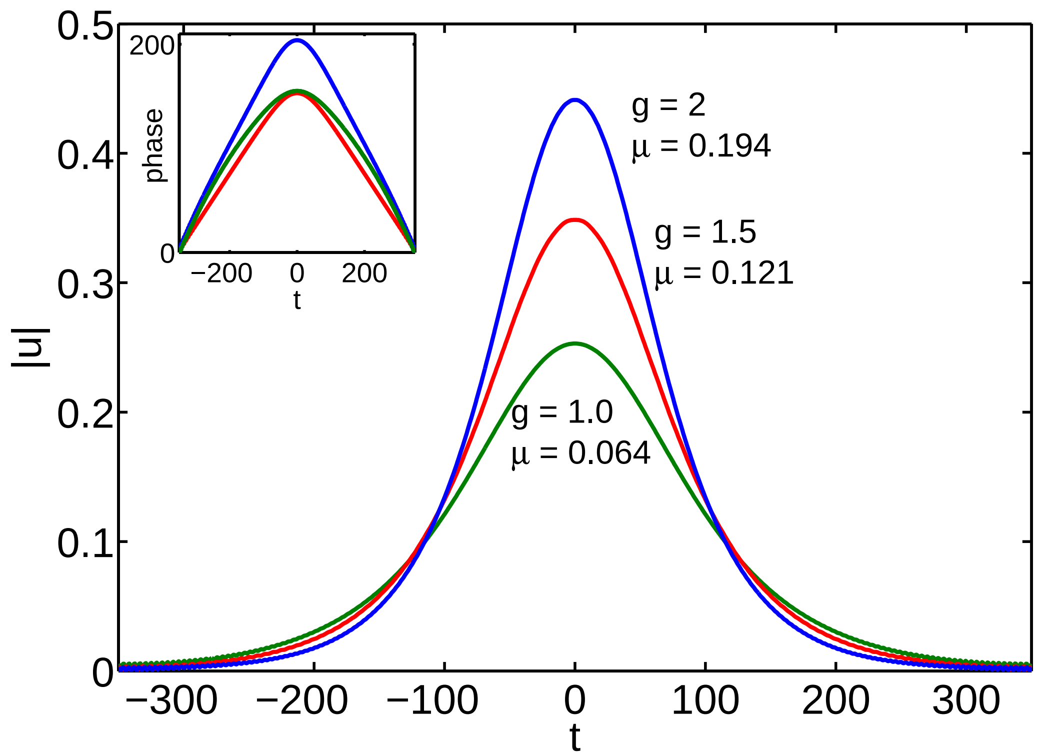
For the PSE eq. with $d_0 < 0; d_0$ constant

$$iu_z - \frac{|d_0|}{2}u_{tt} + |u|^2 u = \frac{ig}{1+E/E_0}u + \frac{i\tau}{1+E/E_0}u_{tt} - \frac{il}{1+P/P_0}u$$

where $E(z) = \int_{-\infty}^{+\infty} |u|^2 \,\mathrm{d}t, P(z,t) = |u|^2,\ \ g, \tau, l, E_0, P_0 > 0$ const.
For $g > g_*$ we find “fat”, strongly phase-chirped solitons.
Typical cases: $E_0 = P_0 = 1, d_0 = -1, \tau = l = 0.1,$ vary g

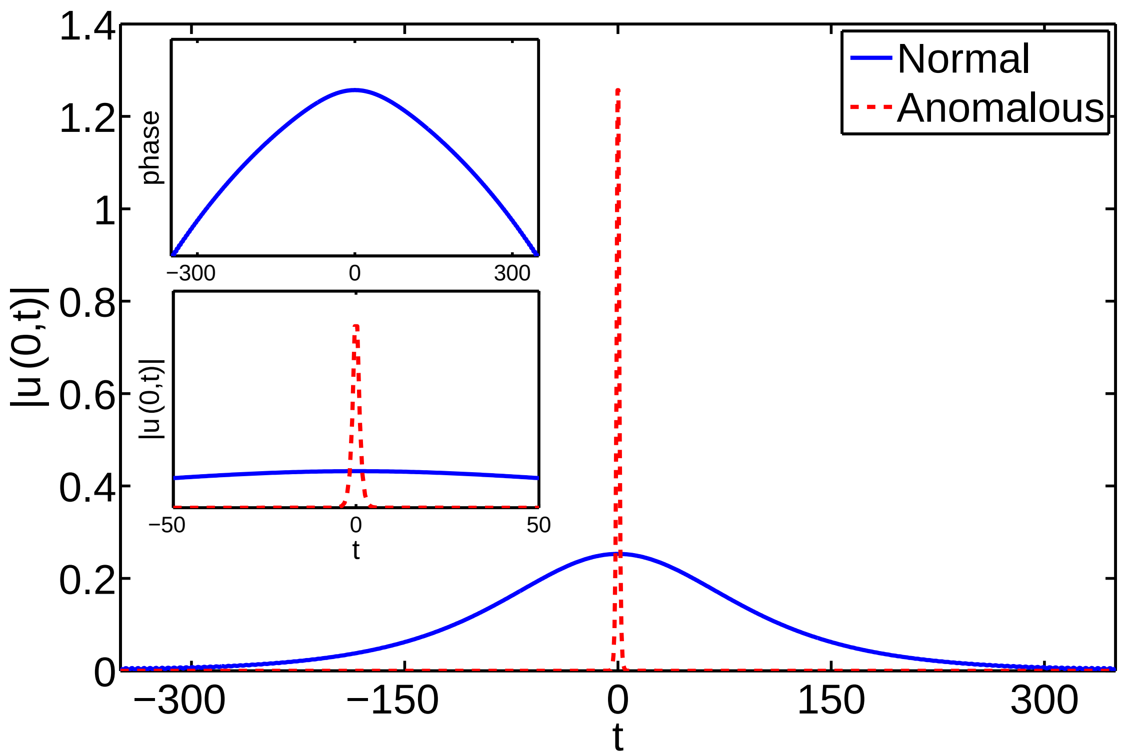
Normal dispersion–solitons

Soliton modes can obtained separately $u(z,t) = u(t)e^{i\mu z}$, $u(\pm\infty) \to 0$; find modes for $g > g_*$



Soliton solutions for different values of the gain parameter g and the corresponding propagation constants: μ; in the inset, the corresponding phases are plotted. Agrees with direct evolution

Normal dispersion-con't



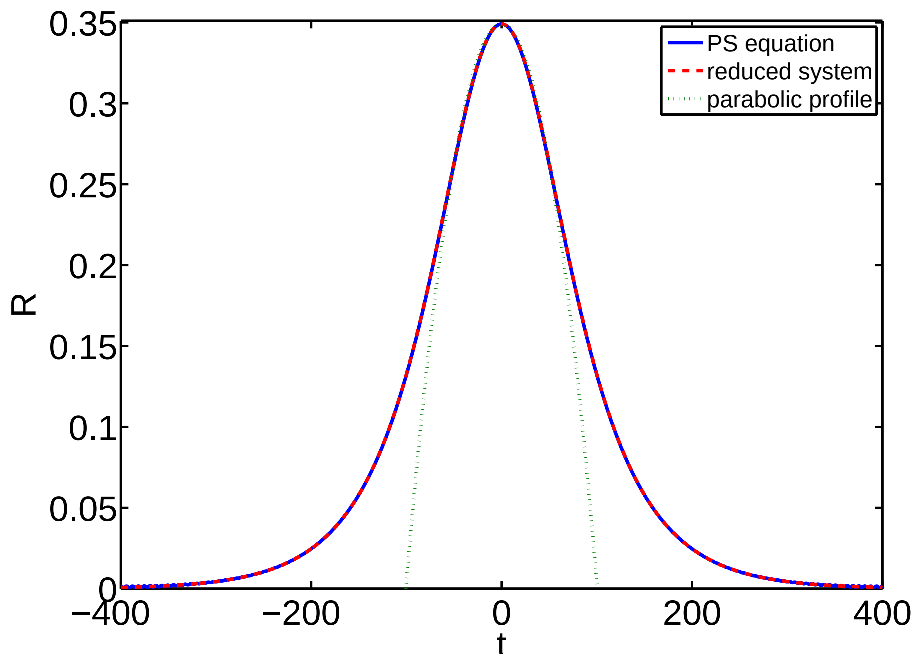
Comparison with the corresponding mode of the anomalous eq. ($g = 1$); in the upper inset the phase of the chirped pulse is shown

Normal dispersion-reduced model

Look for a solution in form: $u(z,t) = R(t)\exp(i\mu z + i\theta(t))$, where R, θ_t are slowly varying. Find

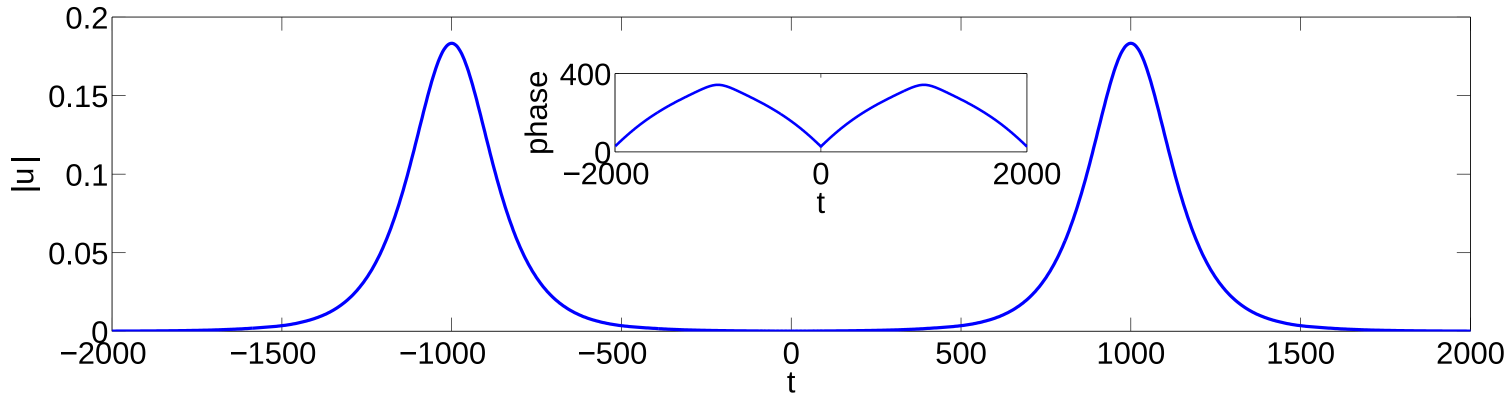
$$\theta_t^2 = \frac{2}{|d_0|}(\mu - R^2), \qquad R_t = F[R, E(R), \mu]$$

$E(R) = \int_{-\infty}^{+\infty} R^2 \,\mathrm{d}t$. Below comparison exact and reduced ($g = 1.5$); a relative parabolic profile is also shown:



Normal Bi-soliton: symmetric

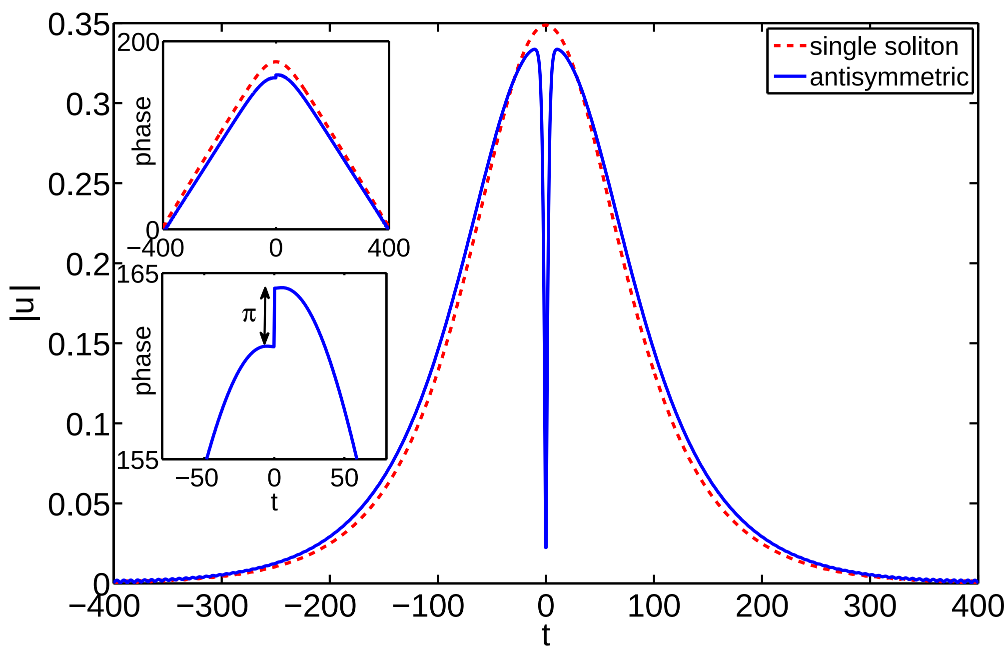
Find higher order strings of solitons in the normal regime;
Some interesting cases: symmetric bi-soliton soliton state:
“in-phase”; find this if the initial distance between peaks:
$\Delta\xi > 9\alpha$ where α is FWHM



Symmetric “in-phase” 2-soliton state of the normal regime
($g = 1.5$); phase chirp is depicted in the inset

Normal Bi-soliton: anti-symmetric

Anti-symmetric bi-soliton; differ in phase by π



Above an anti-symmetric soliton superimposed with the relative single soliton ($g = 1.5$)

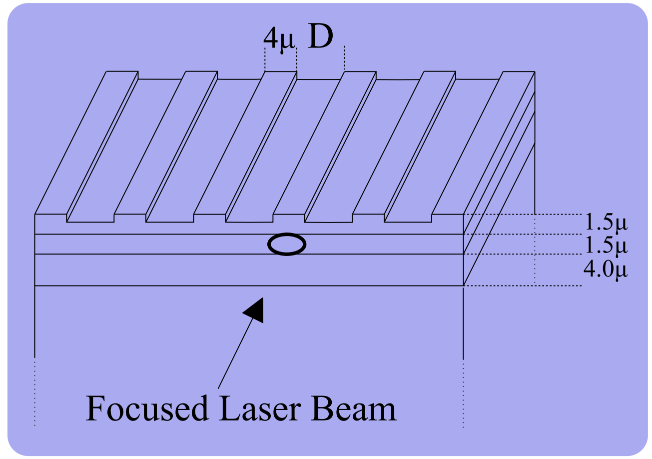
Recently Chong et al OL, 2008, observed an anti-symmetric bi-soliton

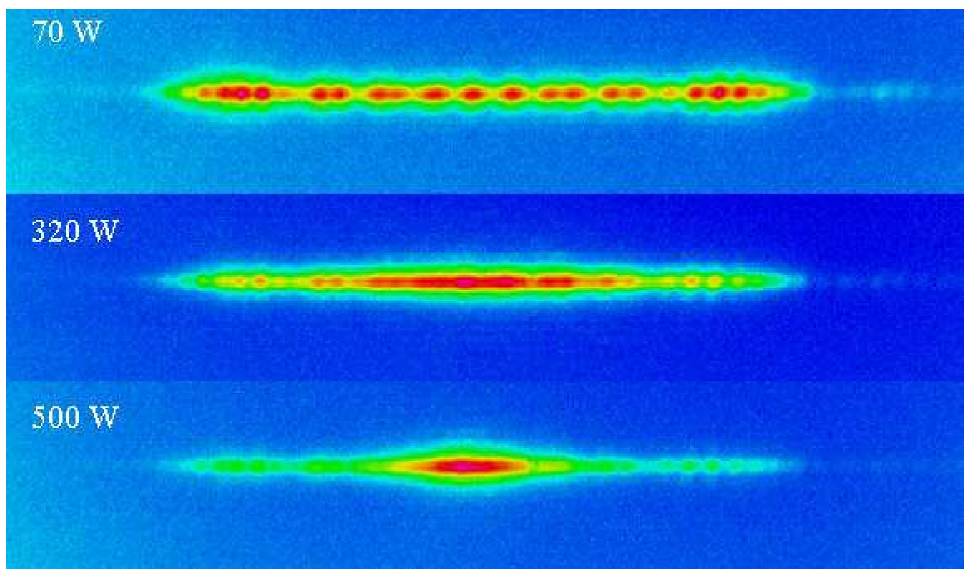
PES–Summary

For PES Eq. (MJA, Horikis, Ilan, Nixon, Zhu; ’08-’09) find:

- Mode-locking occurs for a *wide range* of parameters in constant dispersion and DM systems
- Localized modes, i.e. solitons obtained; they agree with mode-locked states; modes appear to be stable;
- In normal regime pulses are “fat” with phase chirp
- Find mode-locking to strings of solitons; e.g. symmetric and anti-symmteric bi-solitons (latter observed in experiments: OL, 2008)
- solitons numerically/analytically; may describe soliton evolution by perturbation methods
- Refs. MJA, Horikis, Ilan, Nixon, Zhu; ’08-’09

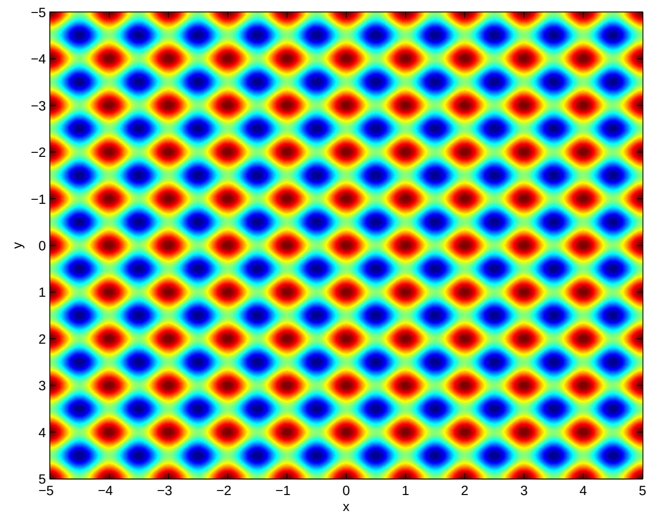
Solitons in Waveguides and Lattices





Power : *a)Low b)Medium c)High* Theory: Christodouledes & Joseph ('88); Exp'ts: Silberberg group ('98...)

Solitons–Periodic Lattice



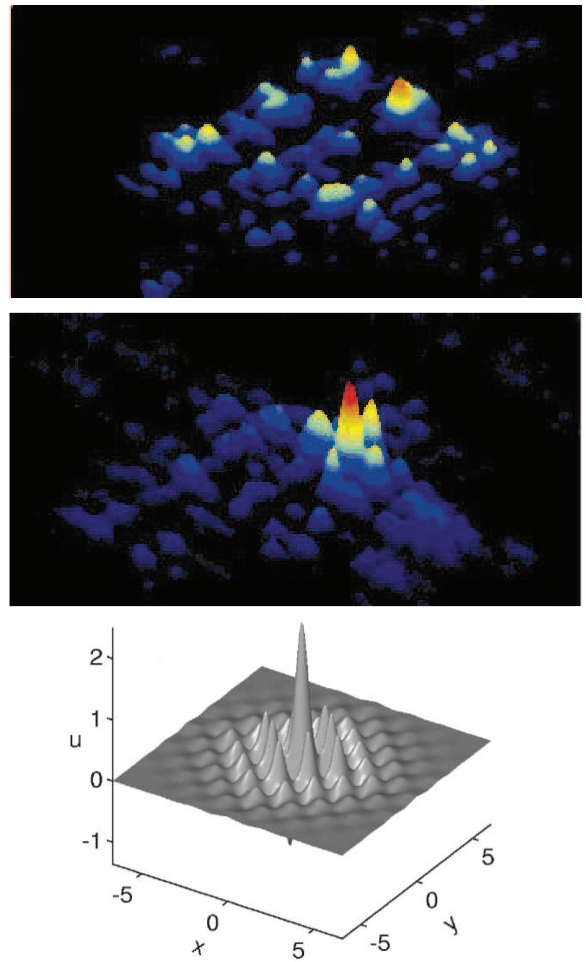
$$V(x,y) = V_0(\cos^2 \pi x + \cos^2 \pi y); V_0 = 1$$

Lattices appear frequently in nature:

- Optical waves on lattice backgrounds
- Photonic crystal fibers
- Bose-Einstein Condensates ....

Most studies consider *periodic* lattices. Many groups have observed and investigated lattice “solitons”

Solitons–2-D Periodic Lattice–con't



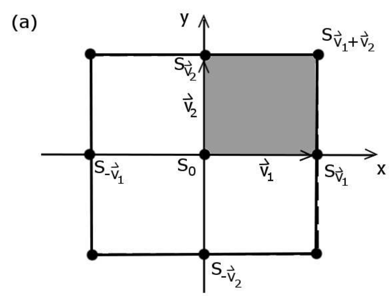
Experiment: Top-middle: Low Power; High Power; Bottom: Computational–Ref. Segev Group

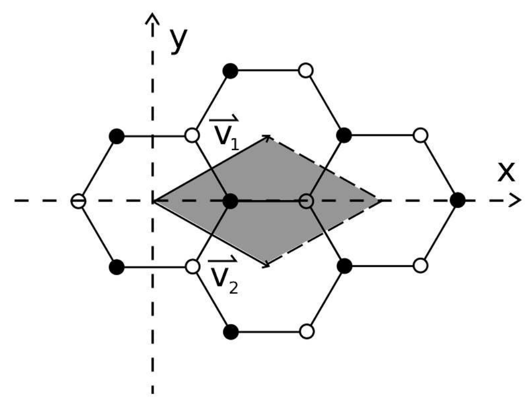
Lattice NLS Eq.

$$i\psi_z + \nabla^2\psi - V(\mathbf{r})\psi + \sigma|\psi|^2\psi = 0$$

where $V(r)$ is a periodic potential defined on a lattice with primitive vectors: $V(\mathbf{r} + m\mathbf{v}_1 + n\mathbf{v}_2) = V(\mathbf{r})$ integers m, n.
Simple lattice: all sites can be constructed from one site.
Non-simple lattices are constructed from more than one site.
Below: typical 2-D lattices

Left: simple rectangular   Right: non-simple HC lattice





Disersion Relation

Dispersion relation–also called band structure obtained from the linear lattice eq.

$$i\psi_z + \nabla^2\psi - V(\mathbf{r})\psi = 0$$

Substituting $\psi = e^{-i\mu z}\phi(\mathbf{r})$: leads to:

$$\mu\phi + \nabla^2\phi - V(\mathbf{r})\phi = 0$$

Bloch modes from: $\phi(\mathbf{r};\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}}U(\mathbf{r};\mathbf{k})$ where $U(\mathbf{r};\mathbf{k})$ has the same periodicity as the potential $V(\mathbf{r}) : \mathbf{v_1}, \mathbf{v_2}$ The dispersion relation $\mu(\mathbf{k})$ has two periods $\mathbf{k}_1$ and $\mathbf{k}_2$, called reciprocal lattice vectors. For each k may solve for U and $\mu(\mathbf{k})$

2-D Periodic Lattices

Tight binding approx. (TBA), $|V| >> 1$

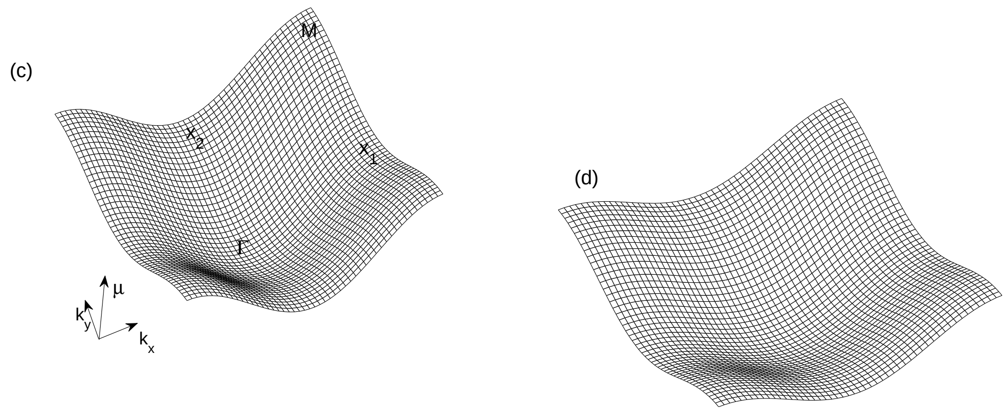
- Simple and non-simple lattices studied widely in physics by TBA; HC lattice: 'graphene'; Wallace 1947 (exp't 2004)
- May find analytical approx. to the dispersion function

$$\mu \approx \mu_0 + \sum_{\mathbf{v}} C_{\mathbf{v}} e^{i\mathbf{k}\cdot\mathbf{v}}$$

where $C_{\mathbf{v}}$ are integrals over wavefunctions and potential

Typical simple lattice dispersion surface

Typical case: square lattice–Lowest band



Left: Numerical; Right: Analytical via TBA

2-D Periodic Lattices

MJA & Yi ’09-’10

- Find discrete and continuous evolution equations of NLS type for simple lattices and nonsimple Honeycomb (HC) lattices (smooth dispersion surface)
- Connect coefficients of the NL evolution system to the dispersion relation of underlying linear Bloch system; dispersion surface can be obtained analytically in TBA
- In simple lattices soliton solutions obtained. In HC lattice (near ‘Dirac points’) find novel systems: nonlinear discrete and continuous Dirac systems which exhibits conical diffraction –a phenomena recently seen in laboratory experiments (Segev group ’07–). Also BEC: (Haddad and Carr ’09)

Discrete NLS eq.– simple lattice

$$\psi(\mathbf{r}, z, Z) \sim \left(\sum_{\mathbf{v}} a_{\mathbf{v}}(Z) \phi_L(\mathbf{r} - \mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{v}} \right) e^{-i\mu z}$$

where $Z = \epsilon z$ and ϕ_L satisfies

$$\left(\nabla^2 - V_L(\mathbf{r})\right) \phi_L(\mathbf{r}) = -E\phi_L(\mathbf{r}),$$

with $V(\mathbf{r}) = \sum_{\mathbf{v}} V_L(\mathbf{r} - \mathbf{v})$. Substitute above into lattice NLS,

multiply $\phi_L(\mathbf{r} - \mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{p}}$ and integrate

$$i\epsilon \frac{da_{\mathbf{p}}}{dZ} + \sum_{<\mathbf{v}>} a_{\mathbf{p}+\mathbf{v}} C_{\mathbf{v}} e^{i\mathbf{k}\cdot\mathbf{v}} + g\sigma |a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0,$$

$< \mathbf{v} >$ is the sum over nearest neighbors and $C_{\mathbf{v}}, g$ are integrals over ϕ_L, V_L.

Continuous NLS eq.– simple lattice

Continuous limit

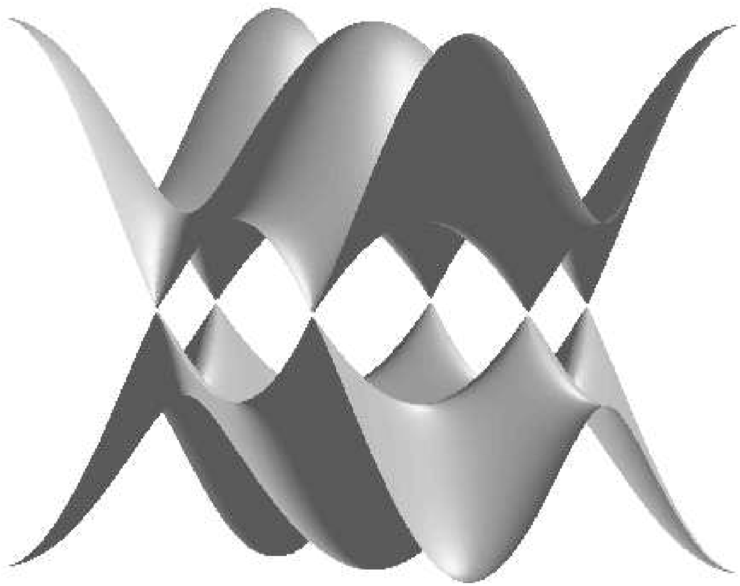
$$a_{\mathbf{p}+\mathbf{v}} \approx a_{\mathbf{p}} + \nu \mathbf{v} \cdot \tilde{\nabla} a + \frac{\nu^2}{2} \mathbf{v} \mathbf{H} \mathbf{v}^T a$$

where $\partial_m = \frac{\partial}{\partial \mathbf{r}_m}$ and $\nabla = (\partial_1, \partial_2)^T$; $\bar{\partial}_m = \frac{\partial}{\partial \mathbf{k}_m}$ and $\bar{\nabla} = (\bar{\partial}_1, \bar{\partial}_2)^T$;
and $\mathbf{H} = \begin{pmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22} \end{pmatrix}$ is the Hessian matrix operator; find:

$$i\epsilon \frac{da}{dZ} + i\bar{\nabla}\mu \cdot \nabla a + \frac{1}{2} \sum_{m,n=1}^{2} (\bar{\partial}_{m,n} \mu) \partial_{m,n} a + g\sigma |a|^2 a = 0$$

HC–Dispersion Surface

Typical dispersion surface, first two bands, below:



See: 1st, 2nd bands can nearly touch at certain isolated points, called Dirac (or diabolical) points. These points can be viewed as edges of a hexagonal region in the reciprocal lattice corresponding to the original potential lattice–i.e. have HC lattice structure in the $\mathbf{k}$ plane.

Dirac Representation

For $V_0 >> 1$, in neighborhood of Dirac point, $k = K$, there is degeneracy:

$$\psi \sim \left(\sum_{\mathbf{v}} a_{\mathbf{v}}(Z)\phi_A(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} + \sum_{\mathbf{v}} b_{\mathbf{v}}(Z)\phi_B(\mathbf{r}-\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{v}} \right) e^{-i\mu z}$$

where $Z = \varepsilon z,\ |\varepsilon| << 1$ and sum $\mathbf{v}$ takes all values $m, n \in \mathbb{Z}$ where $\mathbf{v} = m\mathbf{v}_1 + n\mathbf{v}_2$, and

$$\left(\nabla^2 - V_j(\mathbf{r})\right)\phi_j(\mathbf{r}) = -E\phi_j(\mathbf{r});\ j = A, B$$

here $V(\mathbf{r}) = \sum_{\mathbf{v}}(V_A(\mathbf{r}-\mathbf{v}) + V_B(\mathbf{r}-\mathbf{v}))$

Substitute ψ into lattice NLS eq., multiply

$\phi_j(\mathbf{r}-\mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}};\ j = A, B$ and integrate (Fredholm alternative)

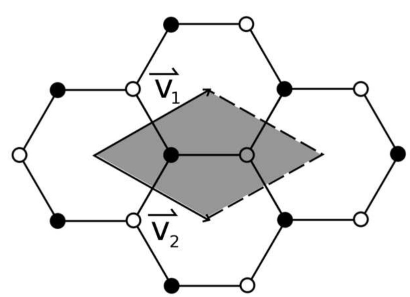
Discrete Dirac

Find, after rescaling (maximal balance: slow evolution, tight binding, nonlinearity) and using the localized properties of ϕ_j

$$
\begin{aligned}
&i\frac{da_{\mathbf{p}}}{dZ} - \mathcal{L}_1 b_{\mathbf{p}} + \mathsf{s}(\sigma)|a_{\mathbf{p}}|^2 a_{\mathbf{p}} = 0 \\
&i\frac{db_{\mathbf{p}}}{dZ} - \mathcal{L}_2 a_{\mathbf{p}} + \mathsf{s}(\sigma)|b_{\mathbf{p}}|^2 b_{\mathbf{p}} = 0, \quad \mathsf{s}(\sigma) = \{1, 0 - 1\}
\end{aligned}
$$

$$
\mathcal{L}_1 b_{\mathbf{p}} = b_{\mathbf{p}} + b_{\mathbf{p}-\mathbf{v}_1} e^{-i\mathbf{k}\cdot\mathbf{v}_1} + b_{\mathbf{p}-\mathbf{v}_2} e^{-i\mathbf{k}\cdot\mathbf{v}_2},
$$

$$
\mathcal{L}_2 a_{\mathbf{p}} = a_{\mathbf{p}} + a_{\mathbf{p}+\mathbf{v}_1} e^{i\mathbf{k}\cdot\mathbf{v}_1} + a_{\mathbf{p}+\mathbf{v}_2} e^{i\mathbf{k}\cdot\mathbf{v}_2}.
$$



Continuous Dirac Eq

Taking the continuous limit ($a_{\mathbf{p}+\mathbf{v}_1} \sim (1 + \mathbf{v}_1 \cdot \nabla + \cdots)a$), find after rescaling:

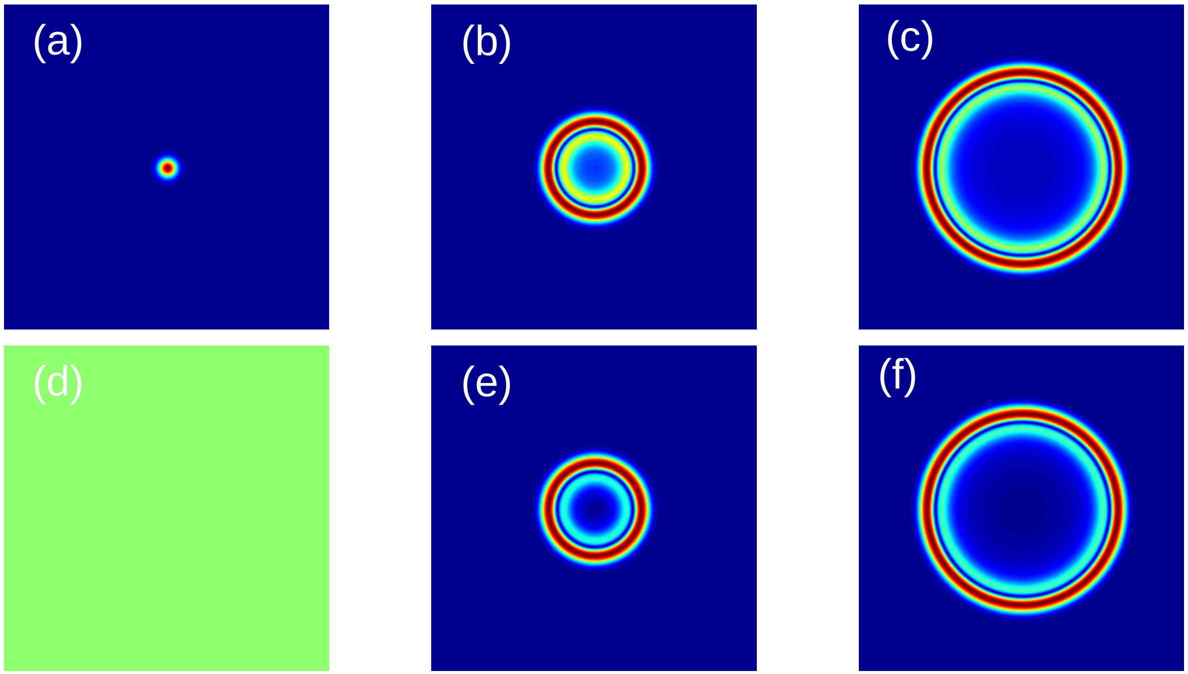
$$i\frac{\partial a}{\partial z} + \partial_{-} b + \mathrm{s}(\sigma)|a|^2 a = 0$$

$$i\frac{\partial b}{\partial z} - \partial_{+} a + \mathrm{s}(\sigma)|b|^2 b = 0$$

where $\partial_{\pm} = \partial_x \pm i\partial_y$, $\mathrm{s}(\sigma) = \{1, 0 - 1\}$

If $\mathrm{s}(\sigma) = 0$, find the linear Dirac system; otherwise find the nonlinear Dirac system. The local linear dispersion relation is: $\omega^2 = k_x^2 + k_y^2$ which approximates the dispersion relation in the neighborhood of the Dirac point.

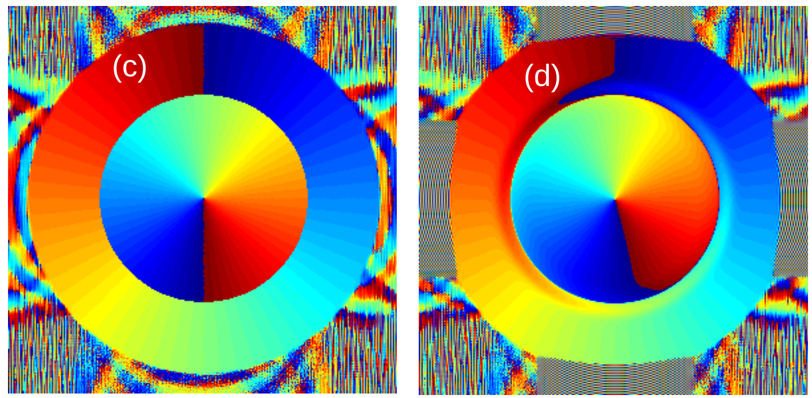
Typical Simulation of Dirac System



Typical simulation of the Dirac equations. The initial conditions: a is a Gaussian and b is zero. (a)-(c) intensities of the a component (d)-(f) intensity of the b component at $z = 0$, 5.4 and 10.8. Find conical diffraction in both components; agrees with direct simulations of the lattice NLS eq.

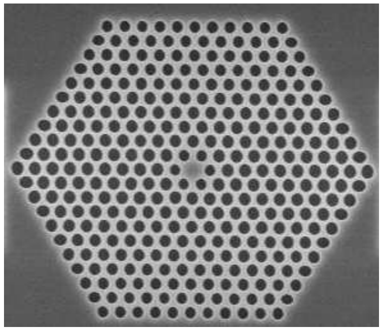
Simulations Phase

Find both and linear and nonlinear Dirac systems exhibit conical diffraction. Difference in phase structure between linear and NL found

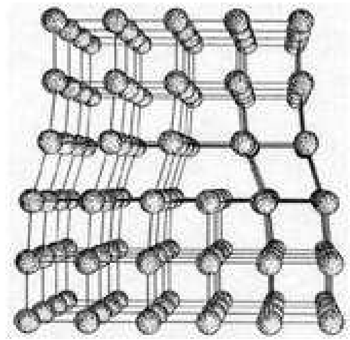


Typical comparison of the phases in the simulations of both linear–left and nonlinear–right Dirac systems. Observe differences in phase structure between linear and nonlinear systems.

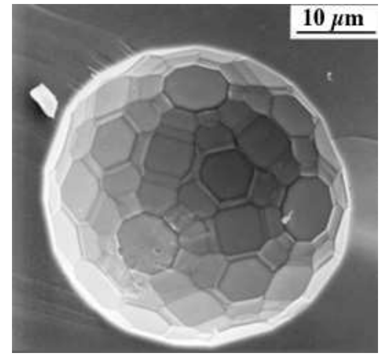
Irregular lattices



vacancy defect



edge−dislocation

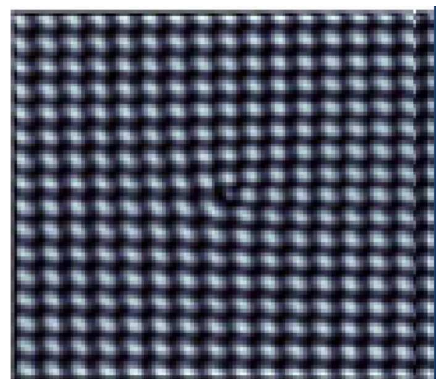


quasicrystal

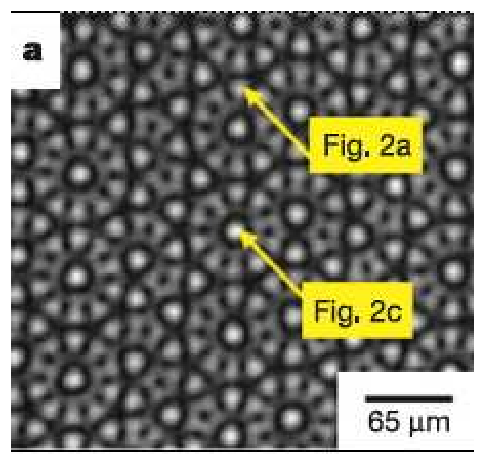
- Also appear widely:
- Point defects (e.g., vacancies)
- Line defects (e.g., edge-dislocations)
- Quasi-crystal structures (e.g. Penrose tiles)

Irregular photonic lattices

Schonbrun & Piestun (2006): introduce various defects and dislocations:



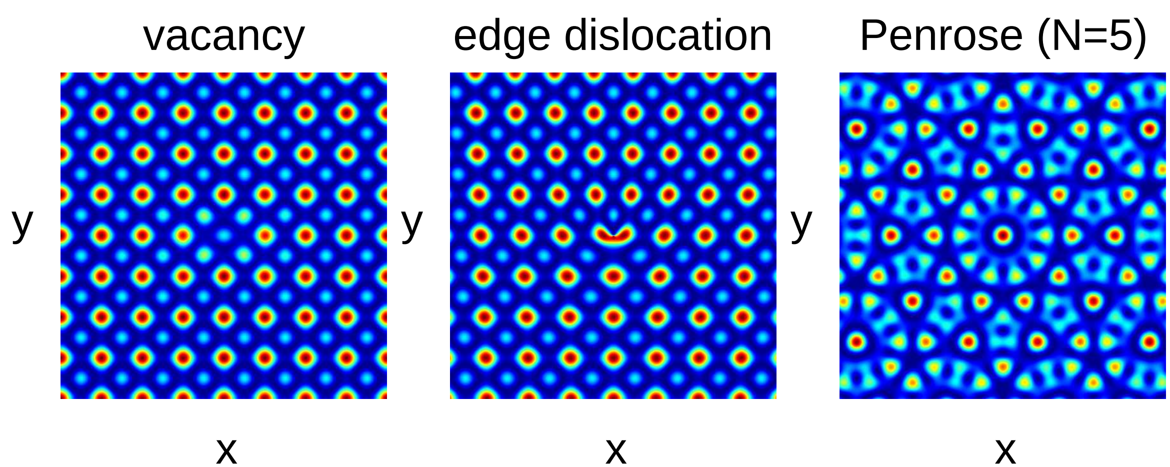
Freedman *et al.* (2006): photonic quasi-crystals:



Localized nonlinear modes–“solitons”

Using numerical methods we can find localized modes on irregular lattice potential backgrounds (MJA, Ilan, Piestun, Schonbrun ’06; MJA, Antar , Birkitas, Ilan ’10).

In particular : vacancy; edge dislocation, quasi-crystal backgrounds:



Localized nonlinear modes

Maxwell + Kerr effect $\Longrightarrow$ **NLS** Eq.

$$i\psi_z + \Delta\psi - V(\mathbf{r})\psi + |\psi|^2\psi = 0$$

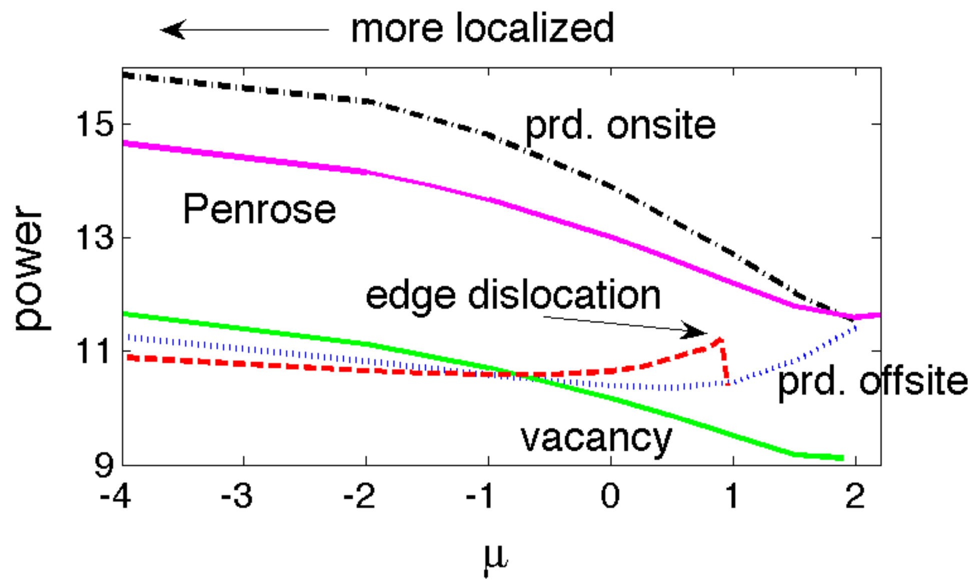
$\psi(\mathbf{r}, z) = f(\mathbf{r})e^{-i\mu z} \Longrightarrow [\Delta + \mu - V(\mathbf{r}) + |f|^2]f = 0$

$f(\mathbf{r})$ = localized, $E = \iint |f|^2\, dx < \infty$

For periodic and aperiodic $V(\mathbf{r})$:

- localized modes (solitons) found numerically; linear problem is useful
- some rigorous theory
- localized modes recently observed in experiments

Energy vs. eigenvalue μ



- typical parameters
- energy lowest with vacancy
- "gap reduced" with edge-dislocation
- periodic-onsite and quasi-crystal modes are "similar"
- $\frac{dE}{d\mu} < 0$ necessary for linear stability (VK)

Conclusion

- Nonlinear waves –in water waves and NL optics
- Water waves:
  - May reformulate as a nonlocal spectral system; find: integral relations, asymptotic reductions: BL,KP,NLS...; can extend theory to multiple fluids.
  - High order series obtained describing large amplitude gravity-capillary solitary waves in 1-d, 2-d
- Nonlinear optics: NLS systems play a central role
  - In communications, and lasers: DM systems important. In lasers PES model has solitons and strings of solitons in anomalous and normal dispersive regimes
  - Photonic lattices: tight binding limit useful; find discrete and continuous NLS and Dirac systems. Latter exhibits conical diffraction–recently observed