U. Arizona Program in Applied Math Qualifying Exam Book List
Last Modified June 15, 2015

Jan 2006 I

1. Elementary ODE. See e.g. [TP85], [Kre02]
2. Counter integration. See Tabor.
3. Basic calculus (if you apply a trig identity)
4. SVD. see [TBI97].
5. Matrix factorizations. See [TBI97].
6. Numerical linear algebra (modified Gram-Schmidt), see [TBI97].
7. Nonlinear dynamical system. See [Per01], [HSD04].
8. Fourier transforms w/ Fubini/Tonelli. See [SS11], [Bra65], [Osg] or Tabor and Flashka
9. Similarity solutions to PDE. See Tabor.
10. Fourier transforms.
11. Dominated convergence theorem.
12. Lagrange interpolation. See [IK12].

Jan 2006 II

1. Newton's method for root finding. See [IK12].
2. Distributions. See Tabor, [Str03] or [GS64].
3. Approximate identities/delta sequence/DCT. See Tabor, Flashka
4. Special solutions of PDE, potential systems. See Tabor for the traveling wave part and [Str14], [Per01], [HSD04] for phase plane analysis.
5. Sturm-Liouville theory, spectral theory. Tabor, [Kre89], and [Fri61].
6. Linear multi-step methods. See [AP98].
7. Real analysis/inequalities. See Flashka, etc.
8. Calculus of variations (no longer covered on the qual, sadly).

August 2006 I

1. ODE/dynamical systems, linear stability analysis, phase plane; chapters 5 and 6 in [Str14], chapter 2 in [Per01], section 4.1 in [HSD04].
2. Complex analysis, Möbius transformation. Chapter 3 in [Nee98].
3. Distributional solutions to differential equations. [GS64] has a small discussion of distributional solutions to ode in chapter 1; [Str03] is a good general reference.
4. This exact problem (and many like it) can be found in [Hab13].
5. ODE systems. For the 'back substitution' approach, see any undergrad ODE book. For the more sophisticated matrix exponential approach, see [Per01], [HSD04].
6. Basic metric spaces. Flaschka is good, or Baby Rudin [Rud64]. I hear good things about [Kap01], but I haven't used it yet.
7. Fourier series. At this level I like [SS11], [Kör89] and [Zyg02], though you can solve this problem by just knowing the definition of a Fourier series on $[0, 1]$. See also the lecture notes and YouTube lectures [Osg].
8. Dominated convergence theorem. Flashka is fine here, or search 'dominated convergence' on math.stackexchange.com to see many many problems worked out.
9. Linear algebra theory. I like [Axl97] and [HK], but there are many other good linear algebra texts. See also [TBI97] lecture 6.
10. See lecture 27 in [TBI97].
11. See lecture 26 in [TBI97].
12. Fake DCT problem; just integrate the function.

August 2006 II

1. Computing with Fourier transforms. See Tabor, [Bra65] or [Osg]. This problem is actually about Gabor frames, aka Weyl-Heisenberg frames, which are connected to the theory of wavelets, time-frequency analysis, and coherent states in quantum mechanics. See, for example, [Chr08].
2. Dynamical systems. See [Per01], [HSD04], or your favorite multivariable calculus textbook (divergence theorem). See also Wikipedia:Reynolds Transport Theorem. This question is unusual for the qual.
3. Complex analysis, specifically branches, Laurent series and contour integrals/residues. See [AF03] or Tabor. Searching 'residue branch cut' on math.stackexchange is also useful.

4. This is a Riemann-Lebesgue problem. See Tabor page 114 or [SW71].
5. Basic real/functional analysis. See Flaskcha or [Kre89].
6. Least squares. See [TBI97].
7. See [TBI97].
8. Linear multi-step methods. See chapter 5 of [AP98].

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Jan 2007 II

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Aug 2007 I

1. Residue calculus. Compute residue with Taylor/Laurent series, carefully. Tabor's book or search math.stackexchange for 'contour integral' or 'residue' to see some nicely worked out examples.

2. Vector calculus. Careful: Green's theorem doesn't immediately apply due to singularity at origin! A good reference is [Kap52].

3. ($\star$) Sturm-Liouville ODE, Green's functions. See Tabor's book or [Fri61]. Because we're on an infinite interval, the Green's function is also easily obtained by Fourier transforming the equation $g'' - g = \delta$. The function g is called the *fundamental solution*, and the Green's function is then 'convolution type' i.e. $K(x, y) = g(x - y)$.

4. ($\star$) Contraction mapping theorem, fixed point interation. See chapter 3 in [IK12]. Also useful here is the multivariable Taylor theorem to first order. Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a. Then, by the definition of differentiable, we must have

$$F(x) = F(a) + J_F(a)(x - a) + o(|x - a|), \qquad (\lim_{x\to a} o(|x - a|) = 0)$$

   The matrix $J_F(a)$ is the Jacobian. The reason why we have $o(|x - a|)$ instead of a quadratic term is that this is a little-oh-estimate, i.e. not an explicit remainder term. Think in analogy to 1 dimension:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(\xi)}{2}(x - a)^2$$

   This quadratic term is an *explicit* ('mean value') remainder term. To estimate its rate of convergence as $x \to a$, we re-arrange to get:

$$\frac{f(x) - f(a)}{x - a} = f'(a) + \frac{f''(\xi)}{2}(x - a)$$

   By the definition of the derivative, if I take $\lim_{x\to a}$, I *must* have the remainder term go to zero, i.e. it must be $o(|x - a|)$.

5. SVD. See [TBI97].

6. Pseudoinverse. Compute $(A^*A)^{-1}A^*$ explicitly, or use SVD. See [TBI97].

7. Real analysis, measure theory. See Flashka.

8. ($\star$) Basic PDE, separation of variables. Tabor and [Hab13]. To answer part (c), note that to be a little more precise we should have written the solution as $u = u_e + u_h$ where u_e is the equilibrium solution (found in part (a) to be $u_e = 2x$) and u_h is a solution of the *homogeneous* problem $u_t = u_{xx}$ with $u(-1, t) = u(1, t) = 0$. The homogeneous problem

will have a solution of the form (obtained from separating variables and performing eigenvalue/eigenfunction/boundary condition analysis):

$$u_h(x,t) \sim \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

In order to have $\partial_t u_h = \partial_{xx}^2 u_h$, we must have $a_n'(t) = -(n\pi)^2 a_n(t)$ and thus $a_n(t) = A_n \exp(-n^2\pi^2 t)$. Thus our final solution $u = u_e + u_h$ takes the form

$$u(x,t) = 2x + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

and hence a typical solution decays exponentially to the equilibrium, $u = u_e = 2x$.

9. Real/Fourier analysis, convolution. See Tabor, Flashka, [Bra65], [Kör89], [SS11],...

10. Principal value distributions. See Tabor and [Str03].

11. ($\star$) DCT for the counting measure. Here is a precise statement of the dominated convergence theorem for abstract measures (see [Fol13], for example):

    Let (X, Σ, μ) be a measure space, and $\{f_n\} \subset L^1(d\mu)$ a sequence such that $f_n \to f$ pointwise and $|f_n| \leq g$ for all n, where $g \in L^1(d\mu)$. Then $f \in L^1(d\mu)$ and

    $$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

    In particular, the above theorem holds in the case of $(\mathbb{N}, 2^{\mathbb{N}}, \#)$, which is the correct measure space to deal with sequences because then $L^1(d\mu)$ is exactly the set of complex-valued sequences x such that $\int_{\mathbb{N}} |f| d\mu = \sum_{n\in\mathbb{N}} |x_n| < \infty$. The index set $\mathbb{N}$ can be replaced with any countable index set like $\mathbb{Z}, \mathbb{Z}^2$, etc and everything still works.

12. Traveling wave solutions to PDE. See Tabor.

Aug 2007 II

1. Linear algebra, spectral radius [TBI97].

2. ($\star$) Finding distributional solutions to singular ODE. Still working on getting a good explanation or reference for this type of question, I'll get back to you. The ansatz is: an equation of the form $a_1(x)y''(x) + a_2(x)y'(x) + a_3(x)$ will have nontrivial weak solutions when $a_1(x)$ has zeros. If the zeros of $a_1(x)$ are isolated, these solutions will always take the form of a derivative or anti-derivative of a sum of delta functions centered on the zeros of a_1. The number of independent distributional solutions is related to the order of the zeros, so for instance $x^2 y''(x) = 0$ has 2 indep. distributional solutions, namely $y(x) = \delta(x)$ and $y(x) = \delta'(x)$.

3. ($\star$) Finding coefficients of a multistep method using a Lagrange interpolating polynomial. In this example, it turns out that we can get the same coefficients by using a Taylor expansion about t_n and matching terms. This *might* in general be true under certain assumptions, but I don't have a proof of this. In general linear multistep methods of a given order are not unique (see question from Aug 2006), so one cannot in general expect two derivations to produce the same coefficients.

4. Subsets of Banach spaces; see Flashka. Two pro tips: 1) a lot of times, the easiest way to prove a set is open or closed is to define an appropriate function $f : X \to Y$, then try to write $A = f^{-1}(B)$ where B is known to be open (resp. closed) in Y. If f is continuous, then A will be open (resp. closed). 2) For compactness, if you're in a metric space (which a lot of the time you are, e.g. normed spaces, inner product spaces), you almost always want to use the definition of *sequential compactness*. It's much easier to work with sequences then open sets, and in a metric space sequential compactness is equivalent to all other notions of compactness.

5. Real analysis trickery. There are probably 20 ways to prove this, just pick a path and it'll probably result in some application of mean value theorem, extreme value theorem, etc.

6. Fourier series trickery, integrating Fourier series term-by-term. See Tabor, [Kör89].

7. Nonlinear dynamical system, change of variables. See Tabor, [Str14], [Per01], etc.

8. Linear multistep, 'stiff decay'. See [AP98]; stiff decay is not an 'industry standard' term, in fact to my knowledge it is only defined in [AP98].

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