Chapter 5: Sampling Distributions

Tuesday (3/10/15): 5.1

- Sampling Distributions
- Sample means and the Law of Large Numbers
- The Central Limit Theorem

Thursday (3/12/15): 5.2, Sampling Distributions for Counts and Proportions
Homework: WebAssign due Friday, Excel assignment posted today, due Tuesday after Spring Break.

Parameters and Statistics

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Notation for samples vs. populations:

Symbol	Use
μ	*Population* mean
$\bar{x}$	*Sample* mean
σ	*Population* standard deviation
s, s_x	*Sample* standard deviation

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– Sample statistics will be treated like random variables, and we already know how to find the distribution (PDF) of a random variable (chapter 4).
– Recall: given a random variable X, its *probability distribution* is a table (for discrete RV's) or function (for continuous RV's) that provides the range of possible outputs of X and their probabilities.

X	2	4	8
$p_X(x)$	1/4	1/2	1/4

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– What is the population standard deviation?

$$\sigma_X = \sum_{i=1}^{5} p_i (x_i - \mu_X)^2 = \frac{1}{5}\left((1-2.5)^2 + \ldots + (5-2.5)^2\right) = 2.25$$

Sample Means and the Law of Large Numbers

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– **We are more interested in the variability of $\bar{x}$ across all possible samples of size 100.**
– In particular, $\bar{x}$ is a random variable, since it is a linear combination of random variables! **What is its distribution?**

Bias, Variability, Unbiased Estimators

Recall a concept from chapter 3:

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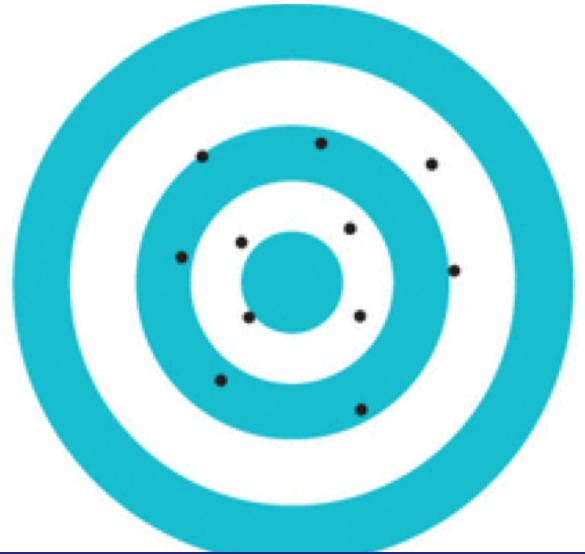

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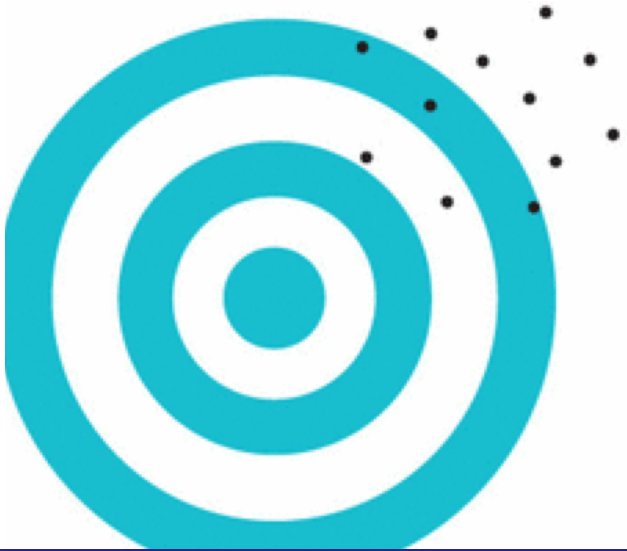

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- In particular, the *population mean and standard deviation* of each X_i is the same:

$$\mu_{X_i} = \mu_X = \mu, \qquad \sigma_{X_i} = \sigma_X = \sigma$$

Another way of saying this: choosing a random sample of 100 *without replacement* from a very large population (e.g. 50,000) will result in the same behavior as if we sampled *with replacement*.

Mean and standard deviation of the sample mean

– Knowing that $\mu_{X_i} = \mu$ for all i, we can compute the mean of $\bar{x}$ (recall that $\bar{x}$ is a random variable!)

$$\mu_{\bar{x}} = E[\bar{x}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\left[\sum_{i=1}^{n} E[X_i]\right] = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n \cdot \mu = \boxed{\mu}$$

So, this says that $\bar{x}$ is an **unbiased estimator of** μ, since the mean of the *sampling* distribution is the same as the mean of the population!

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– We can also compute the sample variance/standard deviation:

$$\sigma^2_{\bar{x}} = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2_{X_i} = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$

so, $\boxed{\sigma_{\bar{x}} = \sigma/\sqrt{n}}$

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- Intuition: averaging (computing $\bar{x}$ over a sample *smooths out* the variability, making sample means less variable than individual observations.
- The question still remains:

What does the distribution of $\bar{x}$ look like?

The distribution of $\bar{x}$

– Using excel, we generate uniform random whole numbers from $\{1, 2, 3, 4, 5\}$ using RANDBETWEEN(1,5). This is like drawing random people from our UA student example - recall that each of 50,000 people had a number between 1 and 5, with the same number of people holding each number.

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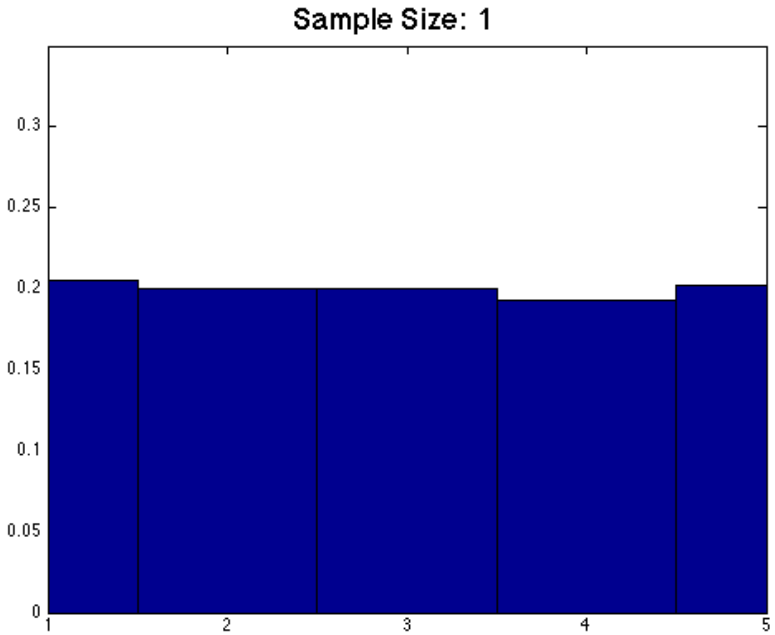
- We compute $\bar{x}$ for different random samples of size n, then make a histogram of $\bar{x}$

The distribution of $\bar{x}$

We know that for samples of size n, $\bar{x}$ has mean μ (same as the population mean) and standard deviation $\sigma/\sqrt{n}$. The distributions look like:



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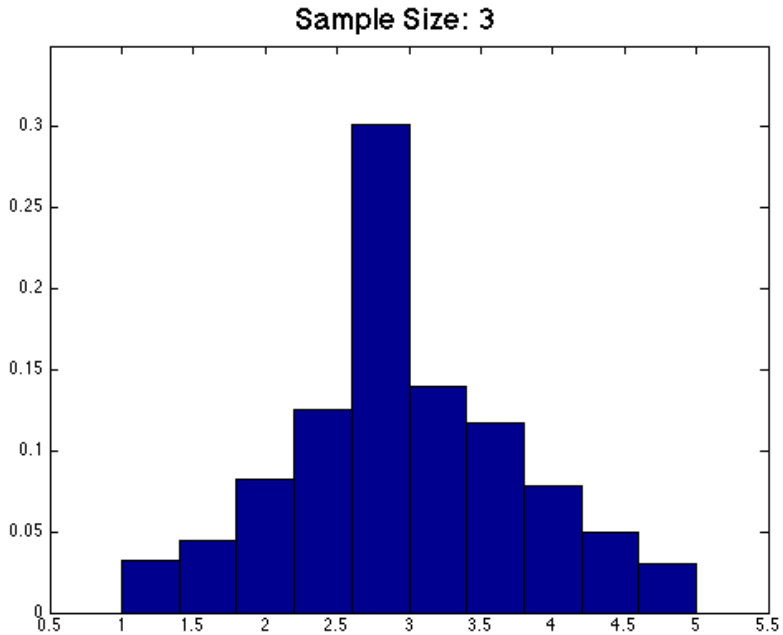

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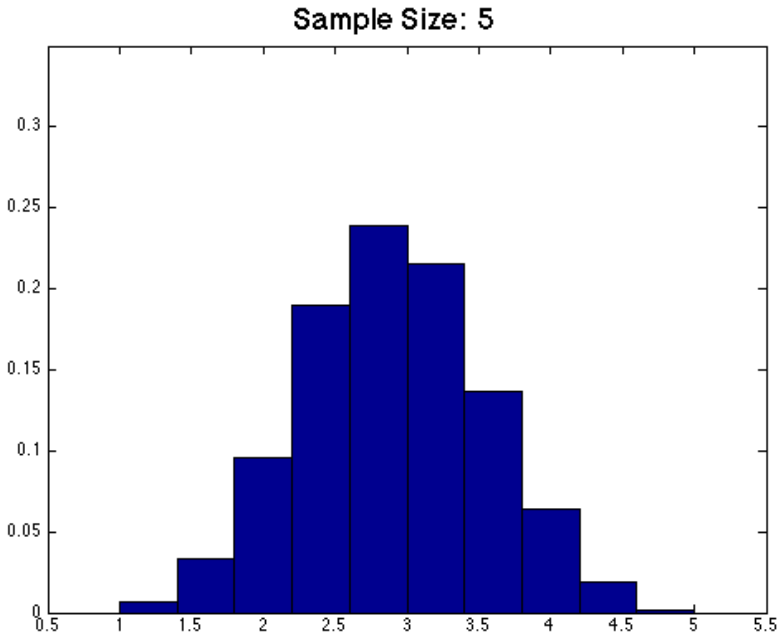

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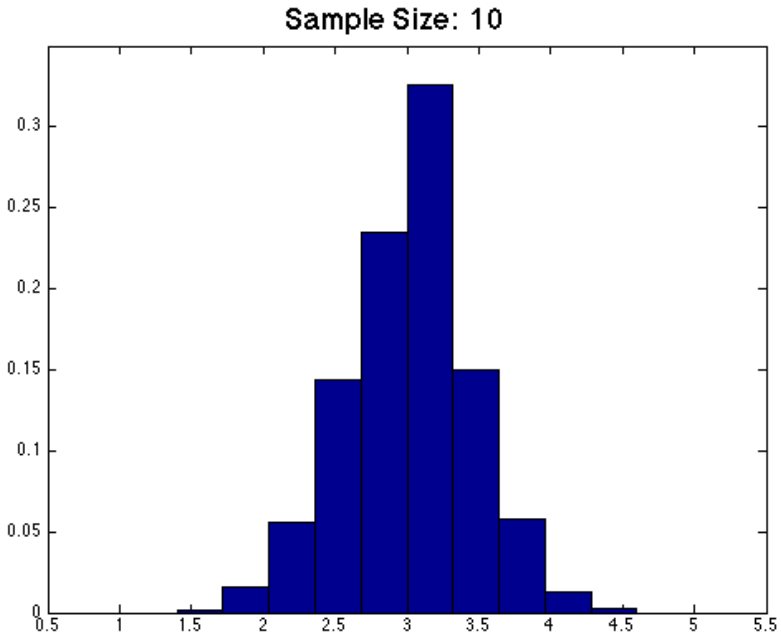

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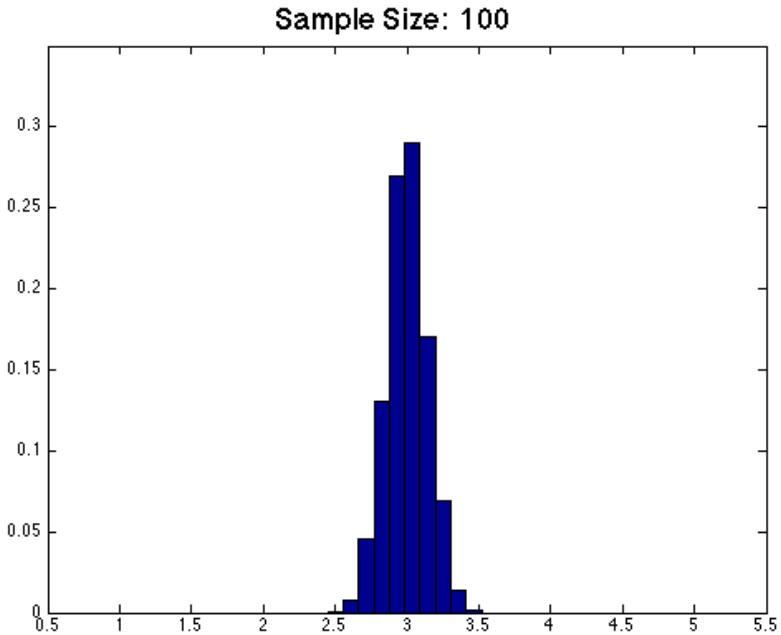

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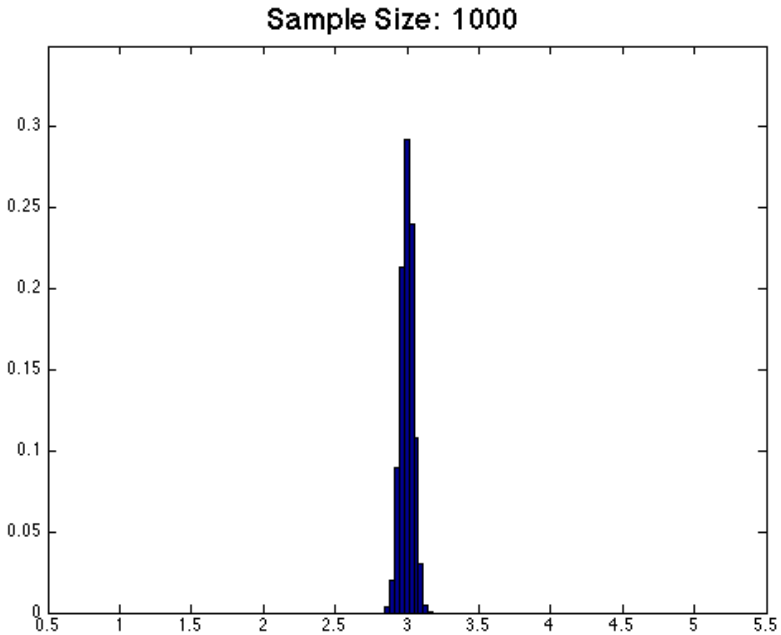

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The Central Limit Theorem

Suppose that $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables. For instance, drawing random samples from a large population, with $n \ll$ population size, each individual shares the same distribution as X, the population parameter. Then, the distribution of

$$\bar{x} = \frac{1}{n}(X_1 + X_n + \ldots + X_n)$$

*is **approximately normal** (unless the X_i are normal) with mean μ_X and standard deviation $\sigma_X/\sqrt{n}$. If the X_i are normal, then $\bar{x}$ is exactly $N(\mu, \sigma/\sqrt{n})$.*

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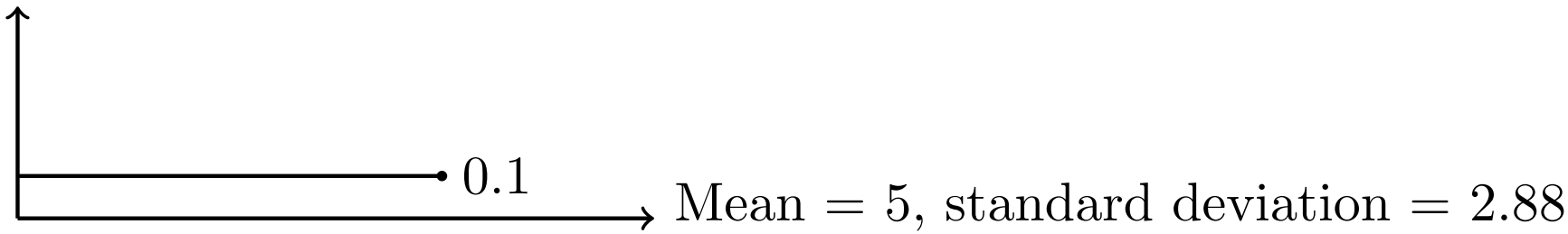
– Note: It does *not matter* what the probability distribution of the variables X_i is! This works with normally distributed RVs, uniformly distributed RVs, and RVs with crazy skewed distributions.

Example

Suppose that we draw random real numbers uniformly from 0 to 10:



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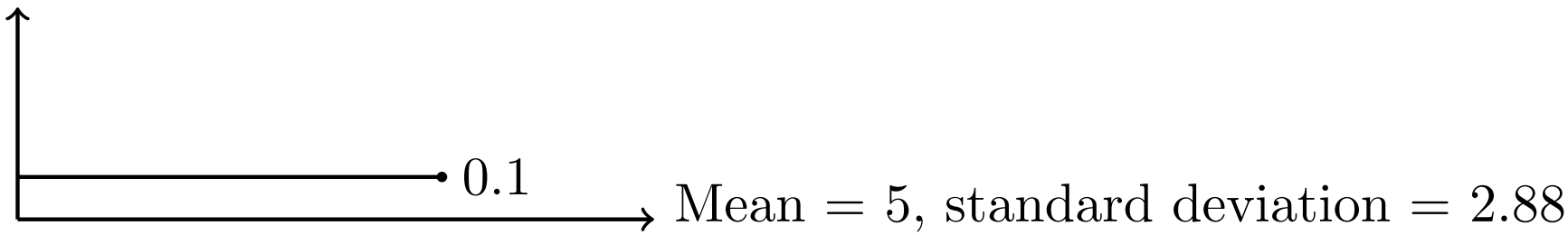

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Answer:

a) Find the area under the PDF between 1 and 2.

b) Using the CLT, we know that $\bar{x} \sim N(5, 2.88/\sqrt{25})$, so we use

normalcdf(1,2,5,2.88/5)

Example

Suppose ACT scores across all incoming freshmen in the US are normally distributed with a mean of 20.8 and a standard deviation of 4.8.

a) Are the given means and standard deviations for the population or for a sample?

b) What is the probability that a randomly selected student had a score higher than 30?

c) What is the probability that 5 randomly selected students have their *average* score higher than 30?

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Answers:

a) These are for the *population*

b) Since this random variable is normally distributed, we can use normalcdf(30,1000,20.8,4.8) = $\boxed{0.028}$ (2 sig figs). As usual, the 1000 is just a 'large' number to indicate that we want all numbers to the right of 30.

Example, cont'd

Solution to part c): by the central limit theorem, we know that

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{5}}\right)$$

Since our original random variable is normal, this is in fact exact - $\bar{x}$ will have *exactly* a normal distribution with mean $\mu = 20.8$ and standard deviation $\sigma/\sqrt{5}$ (5 is the sample size). Thus the probability that 5 randomly selected students has an average score higher than 30 is found with normalcdf(30,1000,20.8,4.8/$\sqrt{5}$) = $\boxed{9.11 \cdot 10^{-6}}$

Review of LLN and CLT

Assumptions:

1) I have n **independent identically distributed** random variables $X_1, \ldots, X_n$, each with mean μ and standard deviation σ.

In most examples, the situation is this: each X_i represents a variable for an individual drawn from a very large population, so that the collection $\{X_1, \ldots, X_n\}$ models a sample of size n. For example, each X_i might be a randomly selected person's age or height or blood O2 level.

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2) The random variables must have **finite mean and standard deviation** (not an issue in this class, but it does come up!)

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Assumptions:

1) I have n **independent identically distributed** random variables $X_1, \ldots, X_n$, each with mean μ and standard deviation σ.

   In most examples, the situation is this: each X_i represents a variable for an individual drawn from a very large population, so that the collection $\{X_1, \ldots, X_n\}$ models a sample of size n. For example, each X_i might be a randomly selected person's age or height or blood O2 level.

2) The random variables must have **finite mean and standard deviation** (not an issue in this class, but it does come up!)

Define the *sample mean* as the random variable

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Review of LLN and CLT

Conclusions:

1) Law of large numbers: $\bar{X} \to \mu$. Loosely speaking we mean that as we collect larger and larger samples, the sample mean 'goes to' the population mean.
2) Central limit theorem: *the variation of* $\bar{X}$ *around* μ *follows a normal distribution.* More precisely,

$$\bar{X} \approx N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

– This means that if we collected many samples of a fixed size n, we would expect the sample means of these samples to be clustered around the actual population mean. The variability decreases with larger samples, **as long as we still satisfy the IID assumption**.

5.2: Repeated Trials and the Binomial Distribution

- A *Bernoulli trial* is an experiment where the outcome is one of two possibilities - i.e. 'yes' or 'no', 'heads' or 'tails', etc.
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- A binary experiment (outcome is one of two options)
- Each 'trial' is independent
- We fix a number of trials at the start
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Example: flip a fair coin 5 times and count the number of heads. What is p? *Answer:* $p = 1/2$ *since the coin is fair.*

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- What is the probability of getting exactly one success in 3 rolls? *Answer: there are 3 different ways to get one success:* SFF, FSF, FFS. *Each has the same probability,* $25/216$. *So, the probability of getting exactly one success is*

$$3\frac{25}{216} = \frac{75}{216} \approx 0.347$$

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Solution: Let X be the number of heads in 100 flips. The probability of getting exactly 20 heads will be

$$P(X = 20) = N 0.4^{20} 0.6^{80}$$

where N is the number of *different ways* to get 20 heads. We can find N using the 'choose' function:

$$\binom{100}{20} \qquad \text{('100 choose 20')}$$

The Binomial Distribution

When we have repeated Bernoulli trials that satisfy the conditions for a binomial setting (independence, fixed number of trials n, fixed probability of success p), the distribution for the number of success is the *Binomial Distribution*:

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Intuitively,

- p^k is the probability of getting k successes
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The number $\binom{n}{k}$ is the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n! = n \cdot (n-1) \cdots 2 \cdot 1$$

Computing Binomial Coefficients

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– You can compute binomial coefficients by hand using Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Example of Binomial Distribution?

– Example: suppose we are going to test a lot of 1000 climbing ropes for strength. We select 5 ropes at random and test them. Suppose the probability of a rope passing a test is 0.999 (i.e. 999/1000 ropes will pass). Define the random variable X to be the number of ropes that **fail** the test in our sample.

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- What is $P(X = 0)$?

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- Try computing this again using a binomial distribution.

$$P(X = 0) \approx \binom{5}{0} 0.001^0 (0.999)^5 \approx 0.99500999$$

Sampling Distribution for a 'Count'

Suppose we sample from a population and count how many individuals satisfy a condition. For example, polling for a yes/no vote on an initiative. We are interested in the distribution of the number of 'successes'.

If we select a simple random sample of size n from a population of size much larger than n, where the probability of success is known to be p, the number of successes k in the sample is ***approximately*** *binomial with parameters p and n:*

$$P(X = k) \approx \binom{n}{k} p^k (1-p)^{n-k}$$

Mean and Standard Deviation of a Binomial RV

The mean and standard deviation of a binomial RV are:

$$\mu = np$$
$$\sigma = \sqrt{np(1-p)}$$

Derivation is tedious, uses the *binomial theorem*:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Approximating Binomial Distributions with Normal

If the number of trials is large compared to either the probability of success or failure (say both np and $n(1-p)$ are bigger than 10), but still small compared to the population size, we have

$$\text{Binom}(n,p) \approx N(np, \sqrt{np(1-p)})$$

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Example: A survey asks a nationwide random sample of 2500 adults their opinion on same-sex marriage. *Suppose* that 55% of Americans [1] actually support same-sex marriage. Estimate the probability that 1400 people **or more** out of the 2500 support same-sex marriage.

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$np = 2500 \cdot 0.55 = 1375 \gg 10$, $n(1-p) = 1125 \gg 10$.

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– Compute $P(X \geq 1400)$ using z-scores or normalcdf:

$$\mu = np = 1375, \qquad \sigma = \sqrt{np(1-p)} \approx 24.875$$

$$P(X \geq 1400) \approx \text{normalcdf}(1400, 10000, 1375, 24.875) \approx 0.157$$

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Sample Proportions

Sometimes, we care more about the *proportion* of a population that satisfies some condition rather than the *number* of people. For example 'what percentage of people drink coffee every day'?

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- Careful to distinguish $\hat{p}$, which is a sample proportion, from p, which is a probability. $\hat{p}$ is like an approximation of p.

Normal Approximation for Proportion

Making the binomial assumption again - i.e. for sample sizes much smaller than the population size (book uses population = 20*sample as a rule-of-thumb), we can use a normal distribution to approximate a proportion:

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Example: suppose we toss a fair coin 500 times. Let X be the number of heads.

a) Find the mean and standard deviation of X

b) Find the mean and standard deviation of $\hat{p}$, the *proportion* of heads out of 500 tosses.

c) Find the probability that between 45% and 50% of the coins showed heads.