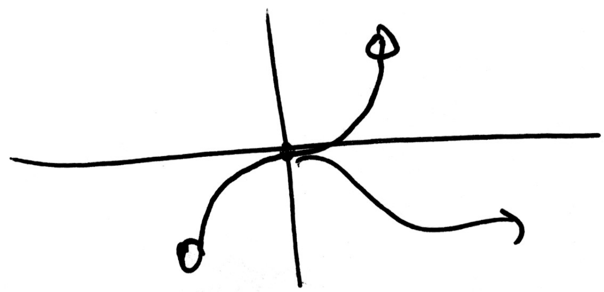
Worksheet 8 Sol^{n}s

1a False. take $f(x) = x^3$, $x \in (-1, 1)$



$f'(0) = 0$

so 0 is a crit pt

but the sign chart is like

f'	$+$	0	$+$
x		0	

Neither min nor max!

1b) TRUE - See Theorem 4.1 in book.
NOTE: Domain is $\mathbb{R}$, & this makes a difference!

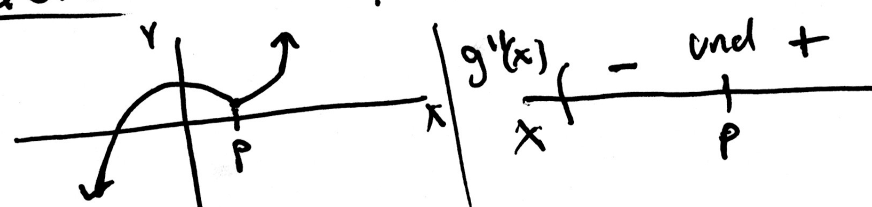
1c TRUE - by Def$^{\underline{n}}$ of global max/min.
~~see~~ e.g. $x=p$ is a global max of $g(x)$
If $g(p) \geq g(x)$ for all x in the Domain of g.

In particular for all x "close" to p.

1d False - take $g(x) = |x|$, $g'(0)$ is undef.
but 0 is a local min.

1e TRUE - This is the Extreme Value Theorem, Theorem 4.2 in the book.

1f FALSE - Inflection points occur @ critical points of the derivative so we could have $g''(x)$ undefined @ $x=p$. e.g.



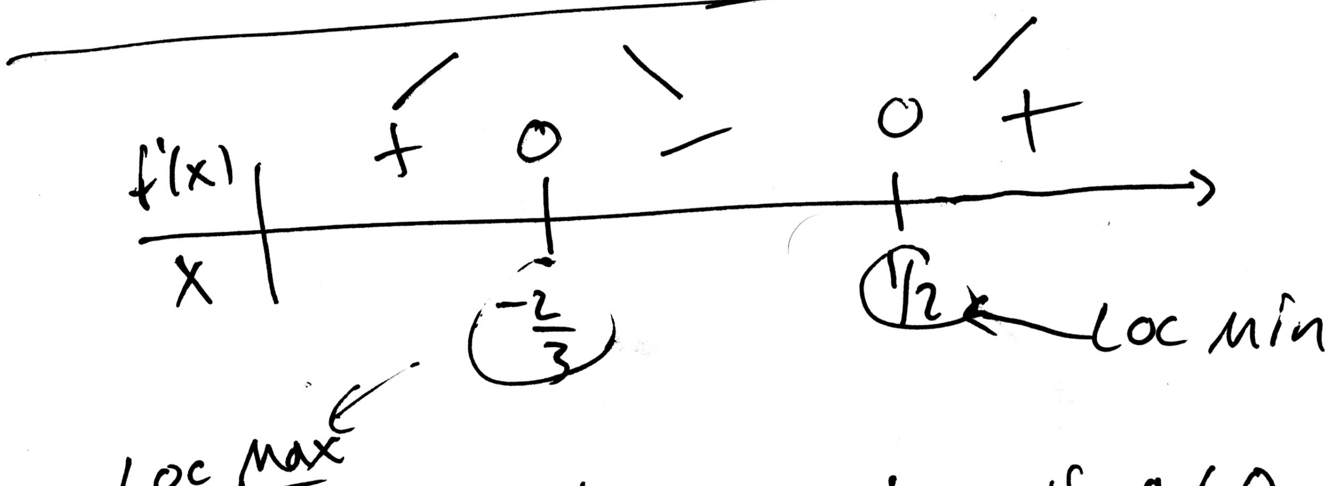
2a) Note: We must consider $a<0$, $a=0$ & $a>0$ separately!

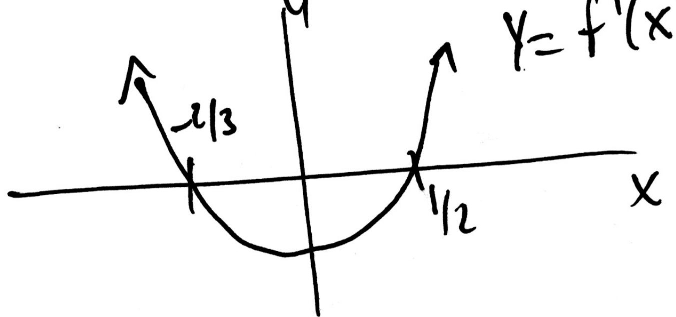
Case 1: $a=0$. Then, $f(x) \equiv 0$, so every point is a critical point, local min & local max.

Case 2: $a<0$

$$f'(x) = a(-12x^2 - 2x + 4)$$
$$= -2a(3x+2)(2x-1)$$

So $f'(x)$ is defined everywhere & $f'(x)=0$ if either $x=-\frac{2}{3}$ or $x=\frac{1}{2}$





$y = f'(x)$ if $a<0$

2a) Cont'd

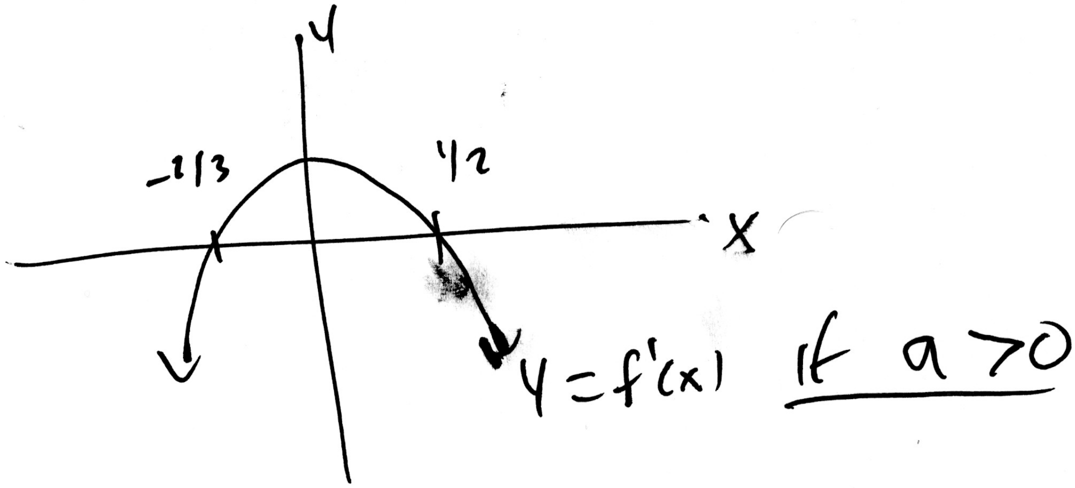
<u>Case 3</u> $a>0$

$$f'(x) = -2a(3x+2)(2x-1)$$

So $f'(x) = 0$ if $x = \frac{-2}{3}$ or $x = \frac{1}{2}$

<u>But</u> sign chart is now:

$f'(x)$	$-$	0	$+$	0	$-$
x		$-2/3$		$1/2$	



<u>2b)</u>

$$f''(x) = a(-24x-2)$$
$$= -2a(12x+1)$$

So $f''(x) = 0$ if $-24x - 2 = 0$

$\Rightarrow x = -1/12$

<u>Sign chart:</u>

$f''(x)$	$+$	0	$-$
x		$-1/12$	

$(a>0)$

2b) Cont'd If $a<0$, sign chart is:

$f''(x)$	$-$	0	$+$
x		$-1/12$	

(Either way, f changes concavity @ $x = -1/12$ so it's an inflect. pt)

3a) Local mins & maxes are either @ crit. pts or endpts since f is defined on a closed interval.

Crit pts: $f'(x) = 9x^2 - 2x + 1$

$f'(x) = 0$ if

$$x = \frac{2}{18} \pm \frac{\sqrt{4 - 4 \cdot 9}}{18}$$

Complex

Since $f'(x)$ has no real roots, $f(x)$ has no crit pts in $[-1, 1]$.

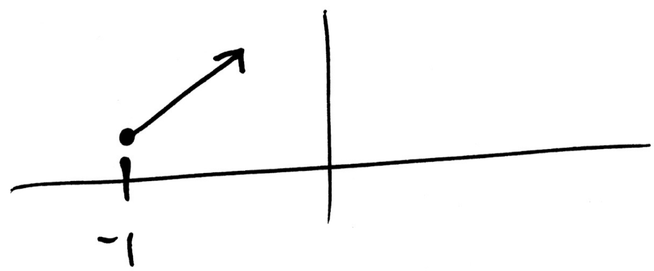
Closed endpts are always local mins/maxes...

3a

$x = -1$ is a local min since $f'(-1) > 0$

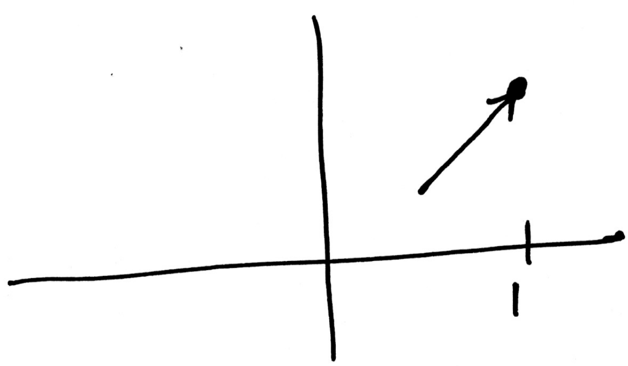
$x = 1$ is a local max since $f'(1) > 0$

3a) Cont'd
To explain, think of the graph: If $f'(x) > 0$ @ left end point:



Function increases away from $x=-1$

If $f'(x) > 0$ @ a right end PT:



increases towards $x=1$.

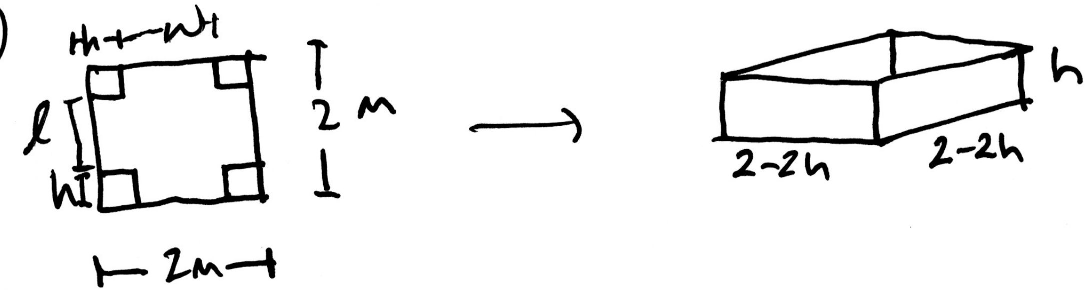
3b $f(x)$ has no local mins/maxes in "interior" $(-1, 1)$, so end PTS are all that's left.

$f(-1) = -3-1-1-1 < 0$ (global min)

$f(1) = 3-1+1-1 > 0$ (global max)

4a

a) We want to maximize the volume of the box

b) 

We vary h, the length of square cut out. The base of the resulting box has dimensions $l = 2-2h$, $w = 2-2h$

c) $$\text{Volume} = l \cdot w \cdot h$$
$$= (2-2h)(2-2h) \cdot h$$

so $$\boxed{\begin{cases} V(h) = (2-2h)^2 \cdot h \\ 0 \le h \le 1 \end{cases}}$$

d) $$V(h) = h(4h^2 - 8h + 4) = 4h^3 - 8h^2 + 4h$$

so $$V'(h) = 12h^2 - 16h + 4$$
$$= 4(3h^2 - 4h + 1)$$
$$= 4(3h-1)(h-1)$$

d) Cont'd

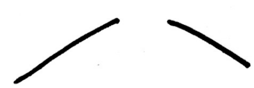
So $v'(h) = 0$ if $h = \frac{1}{3}$ or $h = 1$ → endpoint.

($v'(h)$ is defined everywhere)

e) Technically, the 1st (or 2nd) der. test only applies @ $h = \frac{1}{3}$, since $\mathrm{Dom}(v) = [0, 1]$.

Sign chart:

$v'(h)$	$+$	0	$-$
h		$1/3$	



So $h = 1/3$ is a local max

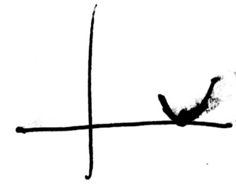
endpts:

$v'(0) = 4 > 0$

So 0 is a local min

$v'(1) = 12 - 16 + 4 = 0$

$v''(1) = 24 - 16 > 0$ Conc. up



Local min

4f $V(0) = 0$

$V(1) = (2-2)^2 \cdot 1 = 0$

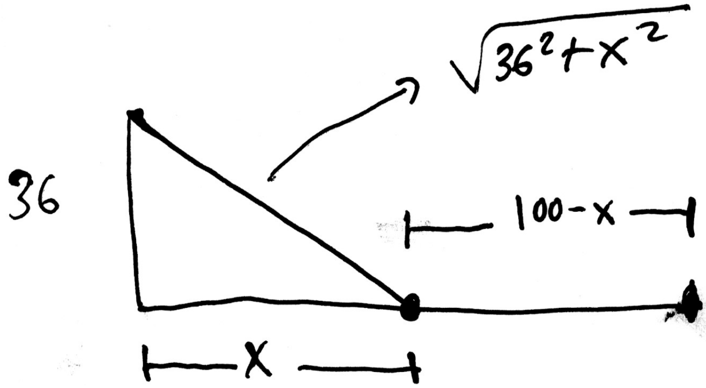
$V(1/3) = \left(2 - \frac{2}{3}\right)^2 \cdot \frac{1}{3} = \left(\frac{4}{3}\right)^2 \cdot \frac{1}{3}$

$= \frac{16}{27}$

$\boxed{\text{Global max @ } (h, V) = \left(\frac{1}{3}, \frac{16}{27}\right)}$

4g ✓

4b) Want to minimize total cost = Cost on Road + cost on side



Cost on road $= 30\sqrt{36^2+x^2}$

Cost on side $= 24(100-x)$

Total Cost $= TC(x) = 30\sqrt{36^2+x^2} + 24(100-x)$

Domain: $0 \le x \le 100$

4b) Cont'd

Critical points:

$$TC'(x) = 15\left(36^2 + x^2\right)^{-1/2} \cdot 2x - 24$$

$$= \frac{2x}{15\sqrt{36^2 + x^2}} - 24$$

$TC'(x)$ is defined everywhere on $[0, 100]$.

$TC'(x) = 0$ if

$$\frac{2x}{15\sqrt{36^2 + x^2}} = 24$$

$$\Rightarrow \quad 2x = 24 \cdot 15\sqrt{36^2 + x^2}$$

$$\Rightarrow \quad 4x^2 = 360^2\left(36^2 + x^2\right)$$

$$\Rightarrow \quad 129596\,x^2 = 167{,}961{,}600$$

$$\Rightarrow \quad x = \sqrt{167{,}961{,}600 / 129596}$$

$$\simeq 36.0$$

Sign chart:

f': $-$ 0 $+$

x: 36

So $x \simeq 36$ is a local min.

4b Cont'd

$$TC(0) = 30 \cdot 36 + 24 \cdot 100 = 3480$$

$$TC(36) = 30 \cdot \sqrt{2 \cdot 36^2} + 24(100-36)$$
$$\approx 3063.35$$

$$TC(100) = 30 \cdot \sqrt{36^2 + 100^2}$$
$$\approx 3188.48$$

So $TC(36) < TC(100)$ & $TC(36) < TC(0)$,

Thus $x = 36$ is the global min on $TC(h)$

4C Silo Prob

Want to minimize total cost = Cost of Cyl + Cost of Dome.

Cost of Dome = 2 $\cdot$ (Cost of sides)

Let $C :=$ The cost of side per ft^2

Cost of Dome = $2 \cdot C \cdot$ (area of Dome)

Cost of Sides = $C \cdot$ (area of sides)

4c) Cont'd

Area of Dome $= 2\pi R^2$

Area of Side $= 2\pi R h$

Side, "unrolled"

So total cost is

$$2 \cdot C \cdot 2\pi R^2 + C \cdot 2\pi R h \qquad (\star)$$

Want in terms of 1 variable (R) & constants (c & V).

$$\text{Volume} = \text{Vol of cyl} + \text{Vol of Dome}$$

$$= \pi R^2 h + \frac{2}{3}\pi R^3$$

Volume is constant - call it V.

$$V = \pi R^2 h + \frac{2}{3}\pi R^3 \quad \text{so}$$

$$\boxed{h = \frac{V - \frac{2}{3}\pi R^3}{\pi R^2}}$$

Plug this into $(\star)$

9c

Finally, TOT. cost in terms of just R & the constants c & V:

$$TC(R) = 4c\pi R^2 + 2c\cancel{\pi} R\left(\frac{V - \frac{2}{3}\pi R^3}{\cancel{\pi} R^{\cancel{2}}}\right)$$

$$= 4c\pi R^2 + 2c\left(\frac{V - \frac{2}{3}\pi R^3}{R}\right)$$

$$= 4c\pi R^2 + \frac{2cV}{R} - \frac{4c}{3}\pi R^2$$

$$= \frac{8c\pi R^2}{3} + \frac{2cV}{R}$$

Domain: $0 \leq R \leq \underbrace{\left(\frac{3V}{2\pi}\right)^{1/3}}$

This is the R where h is zero!

$$TC'(R) = \frac{16c\pi R}{3} - \frac{2cV}{R^2}$$

So $TC'(R) = 0$ if $\frac{16c\pi R^3}{3} = 2cV$

$$\Rightarrow R^* = \left(\frac{6cV}{16c\pi}\right)^{1/3} = \left(\frac{3V}{8\pi}\right)^{1/3}$$

Sign Chart:

$TC'(R)$	$-$	0	$+$
R		$R^* = \left(\frac{3V}{8\pi}\right)^{1/3}$	

(Graph $TC'(R)$ w/ say $c = 1$, $V = 1$ to see this)

$$TC(R^*) = \frac{8c\pi}{3}\left(\frac{3V}{8\pi}\right)^{2/3} + \frac{2cV}{\left(\frac{3V}{8\pi}\right)^{1/3}}$$

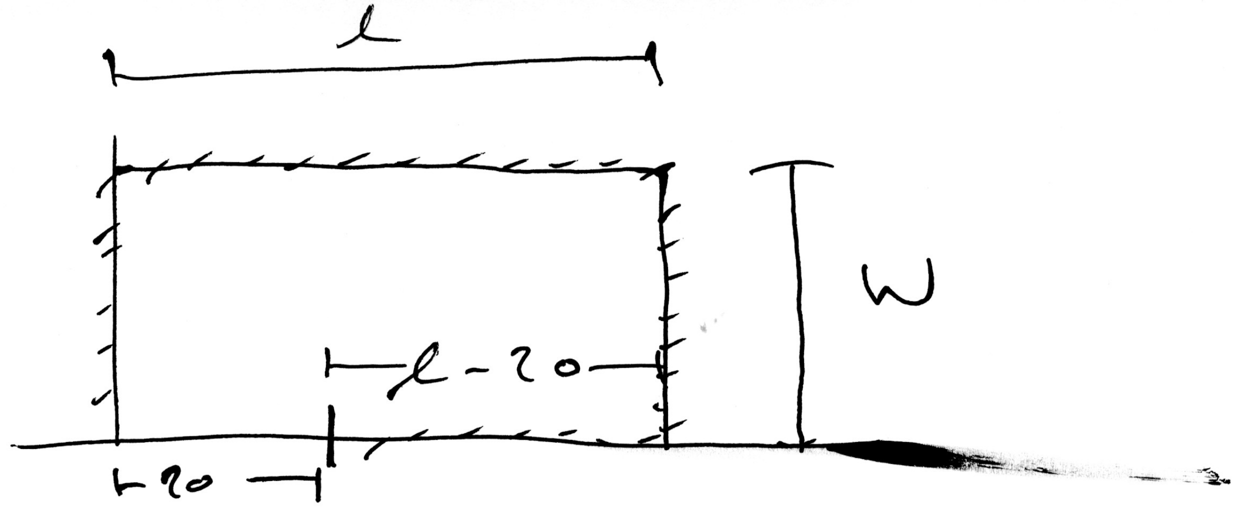
$$TC(0) \text{ "=" } \infty \quad (TC \to \infty \text{ as } R \to 0)$$

$$TC\left(\frac{3V}{2\pi}\right)^{1/3} = \frac{8c\pi}{3}\left(\frac{3V}{8\pi}\right)^{2/3} + \frac{2cV}{\left(\frac{3V}{2\pi}\right)^{1/3}}$$

$$> TC(R^*)$$

So R^* is the global min.

4d



want to maximize area = $l \cdot w$

Total fence $= 2w + l + l - 20$

$= 2w + 2l - 20$

Tot Fence = 1000 so

$2w + 2l - 20 = 1000$

$\Rightarrow \quad 2w = 1980 - 2l$

$\Rightarrow \quad w = 990 - l$

so $\text{Area}(l) = l \cdot w = l \cdot (990 - l)$

$= -l^2 + 990l$

Domain: $20 \le l \le \underbrace{990}_{l \text{ when } w = 0}$

$\text{Area}'(l) = -2l + 990$

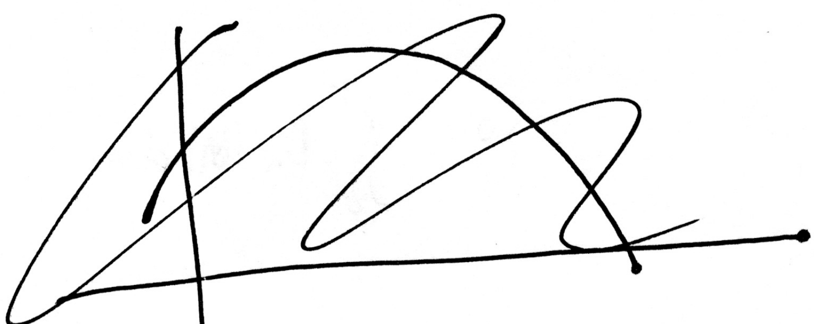
$\text{Area}'(l) = 0$ if $\quad l = \frac{990}{2} = 495$

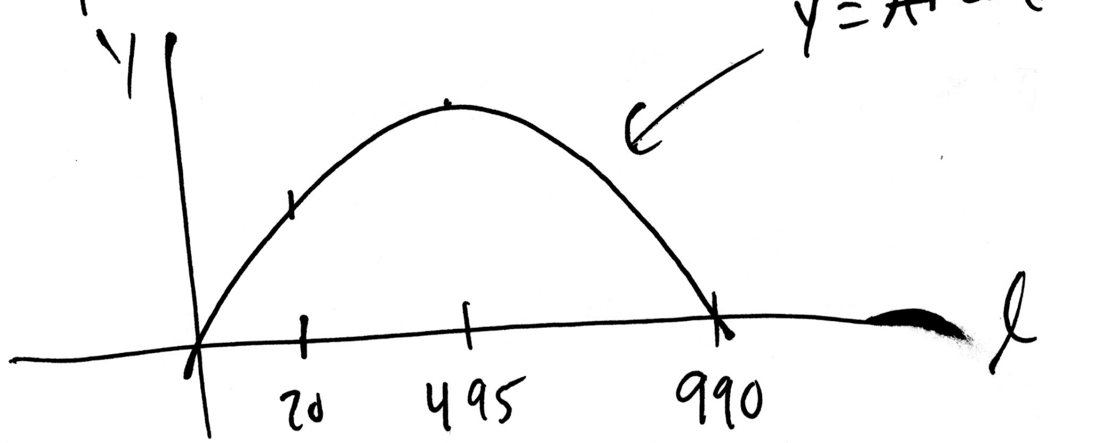
4d cont'd

$$\text{Area}(495) = 495 \cdot (990-495) = 245025$$

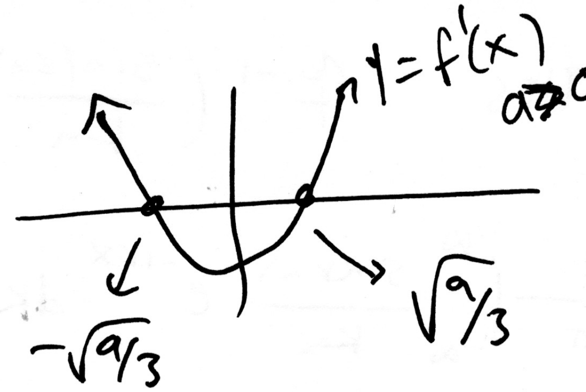
ENDPTS $\begin{cases} \text{Area}(20) = 20 \cdot (990-20) = 19400 \\ \text{Area}(990) = 0 \end{cases}$

So 495 is the global max.





5a)



$$f'(x) = 3x^3 - a$$

so $f'(x) = 0$ if

$$x = \pm\sqrt{\frac{a}{3}}$$ Real since $a > 0$

Sign Chart:

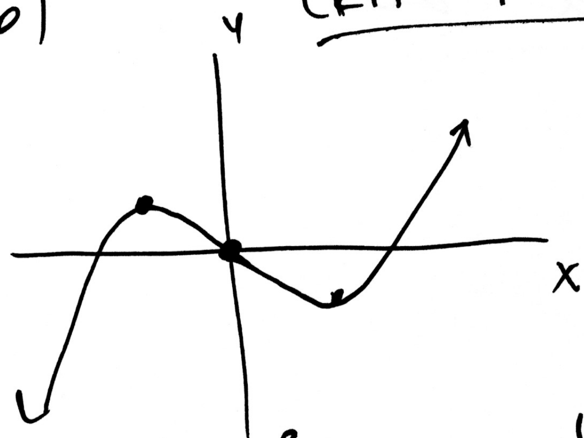
$f'(x)$	$+$	0	$-$	0	$+$
x		$-\sqrt{a/3}$		$\sqrt{a/3}$	

(x, y) coords are thus

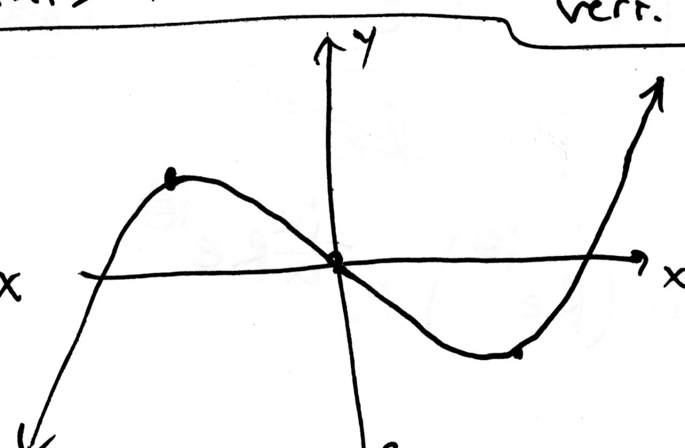
$$(x,y) = \left(-\sqrt{a/3},\ -\left(\frac{a}{3}\right)^{3/2} + a\sqrt{\frac{a}{3}}\right)$$

$$(x,y) = \left(\sqrt{a/3},\ \left(\frac{a}{3}\right)^{3/2} - a\sqrt{\frac{a}{3}}\right)$$

b) CRIT Points ~~ARE~~ are "stretched" vert. & Horiz.



$y = x^3 - x$



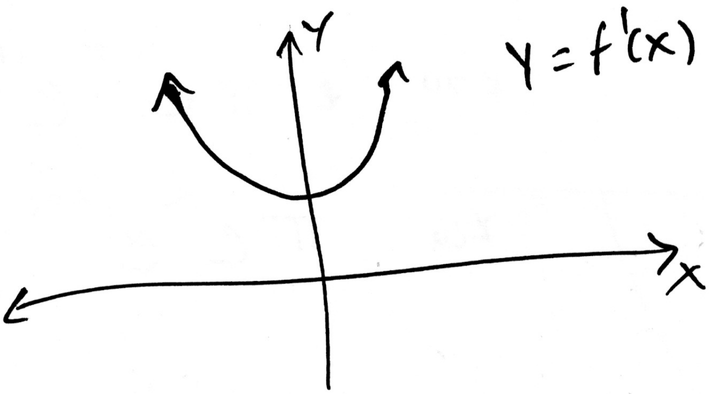
$y = x^3 - 2x$

5c)

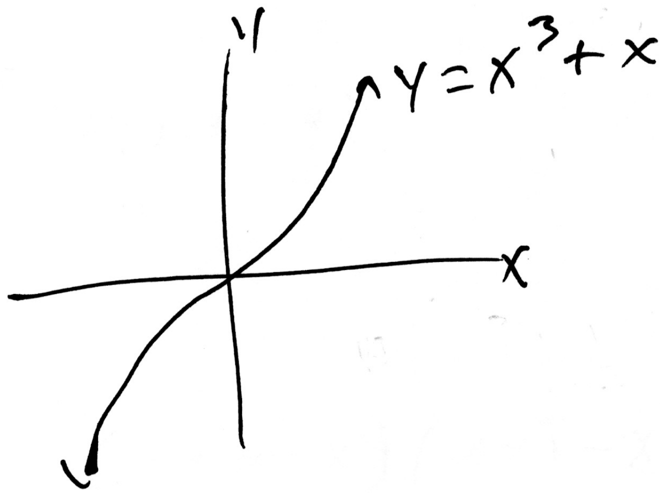
If $a<0$, $x^3 - ax$ has <u>NO</u> crit PTS since

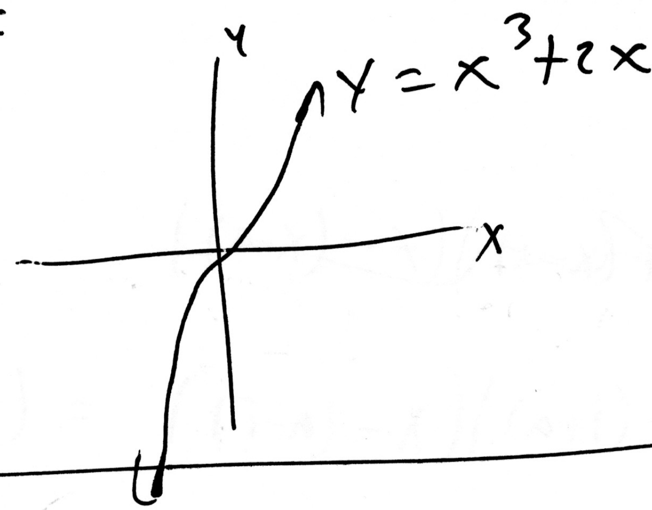
$f'(x) = 3x^2 - a > 0$ for all x.

$y = f'(x)$, $a<0$



d) decreasing a stretches graph - still NO crit PTS





---

e) ~~start~~

$\sqrt{\frac{a}{3}} = 5$ if $a = 75$

so $f(x) = x^3 - 75x$

Check:

$f'(75) = 0$ ✓

$f''(75) > 0$,

so Loc. min.

6a) $V = \frac{4}{3}\pi r^3$ ★

(a) Want $\frac{dV}{dr}$; simply take $\frac{d}{dr}[\,]$ of both sides of ★:

$$\frac{dV}{dr} = \frac{4}{3}\pi 3r^2 \cdot \underbrace{\frac{dr}{dr}}_{1}$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2$$

So if $r = 10$, $\frac{dV}{dr} = 4\pi \cdot 10^2$
$= 400\pi$

(b) Now we want $\frac{dV}{dT}$. Use ★ again:

$$\frac{dV}{dT} = \frac{d}{dT}\left[\frac{4}{3}\pi r^3\right]$$

$$= \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dT}$$

$$\Rightarrow \frac{dV}{dT} = 4\pi r^2 \frac{dr}{dT}$$

So when $\frac{dr}{dT} = \frac{1}{2}$ & $r = 6$,

$$\frac{dV}{dT} = 4 \cdot \pi \cdot 6^2 \cdot \frac{1}{2}$$

$$= 72\pi$$

(c) Now we want $\frac{dr}{dV}$. Use ⊗ again:

$$\frac{d}{dV}[V] = \frac{d}{dV}\left[\frac{4}{3}\pi r^3\right]$$

$$1 = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dV}$$

$$\Rightarrow \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

So when $V = 50$ cm, we have

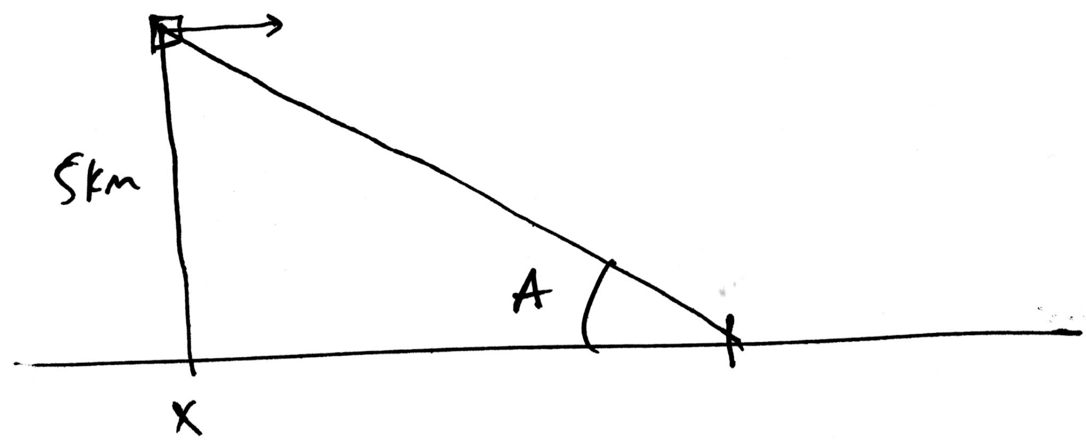
$$r = \left(\frac{150}{4\pi}\right)^{1/3}$$

& so

$$\frac{dr}{dV} = \frac{1}{4\pi\left(\frac{150}{4\pi}\right)^{2/3}}$$

(leave exact)

6b



$X(T)$ ~ Position of Plane @ Time T relative to camera. If $X(T) > 0$, Plane is to the left of Cam, so 450 km/hr means $\boxed{\frac{dx}{dT} = -450}$

Now,

$$\tan(A) = \frac{5}{x}$$

so

$$\frac{d}{dT}[\tan A] = \frac{d}{dT}\left[\frac{5}{x}\right]$$

$$\sec^2 A \frac{dA}{dT} = \frac{-5}{x^2}\frac{dx}{dT}$$

If $A = \pi/3$, $\sec^2(A) = 4$, and $x^2 = \left(\frac{5}{\tan(A)}\right)^2 = 8\frac{1}{3}$

$$4\frac{dA}{dT} = \frac{-5}{x^2} \cdot (-450)$$

$$\Rightarrow \boxed{\frac{dA}{dT} = 67.5 \text{ Rads/Hr}}$$