

## Worksheet 6: Solving Nonlinear Inequalities for Calculus

In the course of learning calculus, it becomes apparent that we might need to solve inequalities of the form

$$f(x) > 0, \quad f(x) < 0, \quad f(x) \leq 0, \quad f(x) \geq 0, \quad f(x) \leq 1, \quad f(x) > g(x), \quad \dots \text{and so on}$$

What does it mean to solve such a thing? We are looking for a *set* of  $x$  such that the inequality is satisfied. This set of  $x$  might be an interval such as  $(0, \infty)$ , or  $[-2, 2)$ , or it might only be a discrete set of points (for instance  $x^2 \leq 0$  if and only if  $x = 0$ ). The set of solutions might be empty! For instance,  $x^2 < 0$  has *no solution* because there are no (real) numbers  $x$  whose square is negative.

Most of the function we encounter in calculus are *piecewise continuous*, meaning they are continuous except for a few jumps - think about  $f(x) = 1/x$ , which consists of two continuous pieces - the piece for  $x < 0$  and the piece for  $x > 0$ . For piecewise continuous functions, we can solve inequalities very easily:

*Three step process for solving inequalities involving piecewise continuous functions*

1. Using addition and subtraction, move all terms to the left hand side of the inequality.
2. Find all zeros and discontinuities of the left hand side.
3. Make a *sign chart* by labeling a number line with your zeros and discontinuities, then checking the sign of the function by evaluating sample points *in between* your zeros and discontinuities.

Why do we care? Solving these precisely will tell us where a function is increasing, decreasing, concave up, and concave down. Knowing this will help us solve modeling problems in chapter 4.

Check out the next page for an example, and the third page for some exercises.

### Examples

Here is an example to illustrate the process: Solve the following inequality:

$$\frac{1}{x} < x^2$$

First, move everything to the left and simplify:

$$\frac{1}{x} < x^2 \Rightarrow \frac{1}{x} - x^2 < 0 \Rightarrow \frac{1 - x^3}{x} < 0$$

Then, we find the zeros and discontinuities of the left side. The function  $f(x) = \frac{1-x^3}{x}$  is zero if and only if the numerator is zero, so  $1 - x^3 = 0$  or  $x = 1$ .  $f(x)$  also has a discontinuity at  $x = 0$  (the function is not defined there).

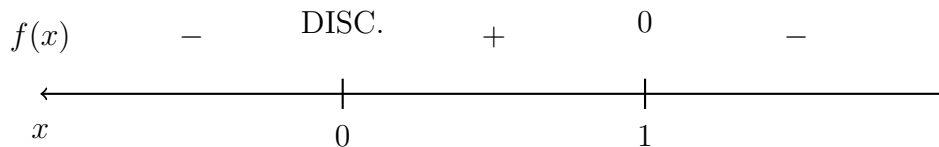
Next, we put these points on a number line, labeling above with either '0' or 'DISC.' to indicate what  $f(x)$  is doing.



Next, we check a point to the left of 0, in between 0 and 1, and to the right of 1 to see what the sign of  $f(x)$  is:

$$\begin{aligned} f(-1) &= \frac{1 - (-1)^3}{-1} = -2 < 0 \\ f(1/2) &= \frac{1 - (1/2)^3}{1/2} = \frac{7/8}{1/2} = \frac{7}{4} > 0 \\ f(2) &= \frac{1 - 2^3}{2} = -\frac{7}{2} < 0 \end{aligned}$$

Finally, we label our sign chart with these values, putting a plus or minus above the intervals we checked:



So, to answer the original question:  $\frac{1}{x} < x^2$  on  $\boxed{(-\infty, 0) \cup (1, \infty)}$

## Exercises

1. Explain why the following line of reasoning fails:

$$\frac{3}{x+2} > x \Rightarrow 3 > x(x+2) \Rightarrow x^2 + 2x - 3 < 0 \Rightarrow (x-1)(x+3) < 0 \Rightarrow \boxed{x \in (-3, 1)}$$

Solve the inequality correctly and rejoice.

2. Let  $f(x) = \sin(2\pi x)$  and  $g(x) = \cos(2\pi x)$ , both with domain  $x \in [-2, 2]$ . Solve

$$f(x) < g(x)$$

3. Let  $Q(z) = \frac{e^z}{z}$ . Find the intervals on which  $Q(z)$  is *increasing*.

4. Let  $f(x) = xe^{x^2}$ .

1. By guess and check, find a function  $F(x)$  such that  $F'(x) = f(x)$ .
2. Where is  $F(x)$  increasing?
3. Where is  $F(x)$  concave down?