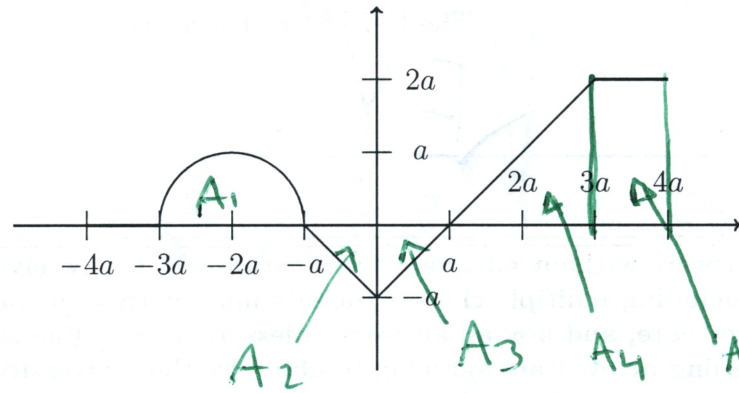
1. (a) For the function f graphed below, compute $\int_{-4a}^{4a} f(t)dt$. The curved piece is a perfect half-circle and $f(x) = 0$ for $x \in [-4a, -3a]$. *Leave your answer exact, in terms of a.*



$A_1 = \frac{1}{2}\pi(a)^2$

$A_2 = -\frac{1}{2}a^2$

$A_3 = -\frac{1}{2}a^2$

$A_4 = \frac{1}{2}(2a)(2a) = 2a^2$

$A_5 = a \cdot 2a = 2a^2$

Big Idea: $\int_a^b f(T)dT$ is the <u>signed area</u> between x-axis and $y=f(x)$

$\int_{-4a}^{4a} f(T)dT = A_1 + A_2 + A_3 + A_4 + A_5 = \frac{1}{2}\pi a^2 - a^2 + 2a^2 + 2a^2$

$= \boxed{\frac{1}{2}\pi a^2 + 3a^2}$

(b) Let $F(t)$ be a function such that $F'(t) = f(t)$, and suppose that $F(a) = -5$. Compute $F(0)$.

Big Idea: $\int_a^b f(T)\,dT = F(b) - F(a)$

So,

$\int_0^a f(T)\,dT = F(a) - F(0)$

$= -5 - F(0) \Rightarrow F(0) = -5 - \int_0^a f(T)dT$

$\int_0^a f(T)\,dT = A_3 = -\frac{1}{2}a^2 \Rightarrow F(0) = -5 - \left(-\frac{1}{2}a^2\right)$

(from above) $= \boxed{-5 + \frac{1}{2}a^2}$

2. Let $g(t) = t^2$. Our goal is to compute a Riemann sum to approximate $\int_0^1 g(t)dt$.

   (a) If we want the difference between the upper and lower sum to be less than or equal to 1/4, how many terms do we require?

$$|\Sigma_R - \Sigma_L| = |f(b) - f(a)|\Delta T \leq \frac{1}{4}$$

$$\Rightarrow |1^2 - 0^2|\Delta T \leq 1/4$$

$$\Rightarrow 1 \cdot \Delta T \leq \frac{1}{4}$$

$\Delta T = \frac{b-a}{n} = \frac{1}{n}$, So

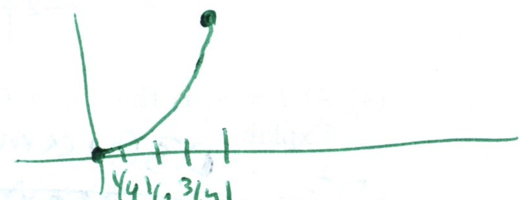
$$\frac{1}{n} \leq \frac{1}{4}$$

$$\Rightarrow \boxed{n \geq 4}$$

   (b) Using the number of terms found in part (a), compute a **right hand** sum. Make sure to list your 'sample points' $t_0, t_1, t_2, \ldots, t_n$. Simplify, but leave your answer exact. *Note: if you were unable to complete part (a), use 6 terms.*

$n = 4$

$$\Delta T = \frac{b-a}{n} = \frac{1}{4}$$



$T_0 = 0 \quad t_1 = \frac{1}{4} \quad T_2 = \frac{1}{2} \quad t_3 = 3/4 \quad T_4 = 1$

RH Sum: $\sum_{i=1}^{4} g(t_i)\Delta T$

$$= \Delta T \left(g(\tfrac{1}{4}) + g(\tfrac{1}{2}) + g(3/4) + g(1)\right)$$

$$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$$

$$= \frac{1}{4}\left(\frac{1+4+9+16}{16}\right)$$

$$= \frac{30}{64} = \boxed{\frac{15}{32}}$$

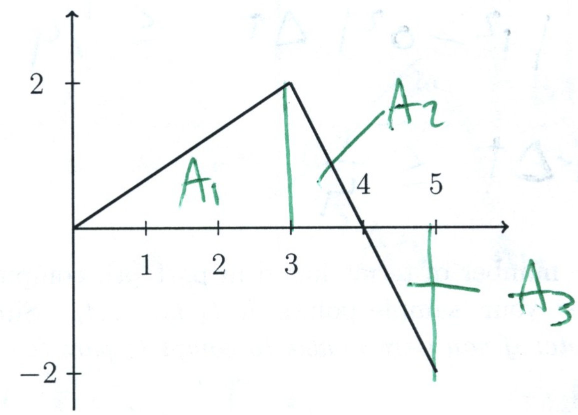
3. The function $v(t)$ given below is the velocity (in meters per second) of an object moving along the x-axis between times $t = 0$ and $t = 5$. Positive velocity indicates the object is moving to the right.

$$v(t) = \begin{cases} \frac{2t}{3} & 0 \le t \le 3 \\ -2(t-3)+2 & 3 < t \le 5 \end{cases}$$

$A_1 = 3$

$A_2 = 1$

$A_3 = -1$



(a) At $t = 5$, is the object to the left of, to the right of, or at its starting position? Explain.

~~Distance traveled~~ Displacement from $T=0$ to $T=5$

is $\int_0^5 v(t)\,dt = A_1 + A_2 + A_3$

$= 3 + 1 - 1 = 3$ meters

So object has moved to the <u>Right</u>

(b) At $t = 4$, how far is the object from its starting position?

~~Dist. travel~~ Displacement from 0 to 4 is $\int_0^4 v(t)\,dt = A_1 + A_2 = 4$ meters

to the right

4. Define the *logarithmic integral* funciton li(x) as follows:

$$\text{li}(x) = \int_0^x \frac{1}{\ln(t)}\,dt, \quad 0 < x < 1$$

Compute $\frac{d}{dx}\left[\text{li}(\sqrt{x})\right]$

$$\frac{d}{dx}\left[li(\sqrt{x})\right] = li'(\sqrt{x}) \cdot \frac{d}{dx}\left[\sqrt{x}\right]$$

$$= \frac{1}{\ln(\sqrt{x})} \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}\,\ln(\sqrt{x})}$$

Big IDea:

$$\frac{d}{dx}\left[\int_0^{g(x)} f(t)\,dt\right] = f(g(x))\,g'(x)$$

Since $\int_0^x f(t)\,dt$ is an anti-Derivative of $f(x)$, IE

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x).$$

5. (a) Find the *general* solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1}$$

Hint: don't simplify the right hand side!

$$\boxed{y(x) = \ln|x+1| - \ln|x-1| + C}$$

Big Idea: general sol^{n} to

$$\frac{dy}{dx} = f(x) \text{ is}$$

$$y(x) = \int f(x)\,dx = F(x) + C$$

where $F(x)$ is any anti-der of $f(x)$.

(b) Now, suppose that $y(0) = 1$. Find the *particular* solution for this 'initial value'.

$$1 = y(0) = \ln|1| - \ln|-1| + C$$

$$= 0 - 0 + C$$

so $C = 1$

Can also simplify to

$$y = \ln\left(\frac{|x+1|}{|x-1|}\right) + 1$$

$$\boxed{y(x) = \ln|x+1| - \ln|x-1| + 1}$$

6. Suppose $\int_{-1}^{2} g(x)dx = A$, $\int_{-1}^{2} f(x)dx = B$, $\int_{2}^{5} f(x) = 0$, and $\int_{2}^{8} f(x)dx = 1$. Find the following:

(a) $\int_{5}^{-1} f(x)dx$

$$= -\int_{-1}^{5} f(x)dx = -\left(\int_{-1}^{2} f(x)dx + \int_{2}^{5} f(x)dx\right)$$

$$= -(B+0) = \boxed{-B}$$

(b) $\int_{5}^{8} f(x)dx$

$$= \int_{2}^{8} f(x)\,dx - \int_{2}^{5} f(x)\,dx$$

$$= 1-0 = \boxed{1}$$

(c) $\int_{-1}^{2} (-f(x) - g(x) + 1)dx$

$$= -\int_{-1}^{2} f(x)dx - \int_{-1}^{2} g(x)dx + \int_{-1}^{2} 1\,dx$$

$$= \boxed{-B - A + 3}$$

Area of a rect, w/ Length 1 width

7. Compute the following antiderivatives.

(a) $\displaystyle\int \left[3^y + a^2 + \frac{1}{y}\right] dy$

$$= \int 3^y dy + \int a^2 dy + \int \frac{1}{y} dy$$

$$= \frac{3^y}{\ln(3)} + a^2 y + \ln|y| + C$$

(b) $\displaystyle\int \frac{3}{1+t^2} dt$

$$= 3 \int \frac{1}{1+t^2} dt$$

$$= 3 \arctan(t) + C$$